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CS 184 | FOUNDATIONS OF COMPUTER GRAPHICS | Professor Brian A. Barsky

## Homogeneous Coordinates

*Homogeneous coordinates* are often used in computer graphics to provide a flexible and uniform representation of points in space. We already saw a special form of homogeneous coordinates, without explicitly referring to them as such, when developing the matrix representation of the simple transformations. Recall that we could not form a matrix for translation until we added an extra row and column to the matrix as well as a corresponding extra element to the row vector representing a point in space. This extra element is sometimes called the *homogeneous component*. In the case of translation, the homogeneous component is now restricted to be "1". Homogeneous coordinates, in general, have a homogeneous component that is free to assume any value. Given points and operations in a space of a certain dimension, the homogeneous coordinates represent this in a space having a dimension one greater than that of the original space. For example, points in three dimensions have a homogeneous representation with four-element points and 4x4 matrices.

To gain a better understanding of homogeneous coordinates, we will begin by considering the homogeneous representation of a point in one dimension. To distinguish between the non-homogeneous and homogeneous representations of the same point, we will use a "*" to denote the nonhomogeneous representation. Let x* be a point in one dimension. Mathematically, we refer to this as a point in $\mathbf{E}^1$, where **E** is for "Euclidean", and the "1" denotes a dimension of 1. We can then associate with every x* a *homogeneous coordinate pair* (x, w) , such that x* = x/w. This representation is not unique since only the *ratio* of the coordinates is determined. The redundancy can be seen in that any non-zero multiple of all coordinates in the homogeneous representation of a point yields another homogeneous representation of the same point.

It is interesting to investigate a geometrical interpretation of the homogeneous coordinate pairs (x, w). We can arrange the equation for x* given in the preceding paragraph as

$$w = \frac{1}{x^*} x .$$

This demonstrates that the set of all homogeneous points (x, w) that represent the same point x* in $\mathbf{E}^1$ is a *line* that passes through the origin and through the point (x*, 1), as shown in Figure 1. The point in $\mathbf{E}^1$ can be thought of as a projection of the two-dimensional homogeneous representation onto the line described by w = 1 by the line that passes through the origin and through the point (x*, 1) as in Figure 1.

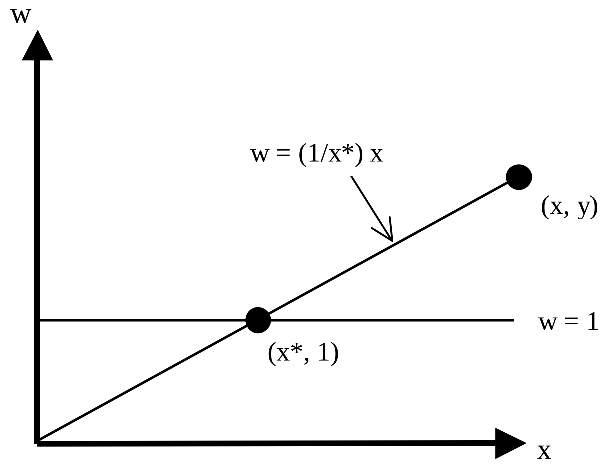


**Figure 1.** Homogeneous representation of $\mathbf{E}^1$.

Having explored the case of a point in one dimension, let's now consider a point in $\mathbf{E}^2$. Analogous to the case of a point in $\mathbf{E}^1$, a point in $\mathbf{E}^2$ can be seen to be the projection of a three-dimensional homogeneous representation onto the plane described by w = 1 by the line that passes through the origin and through the point (x*, y*, 1), as shown in Figure 2.

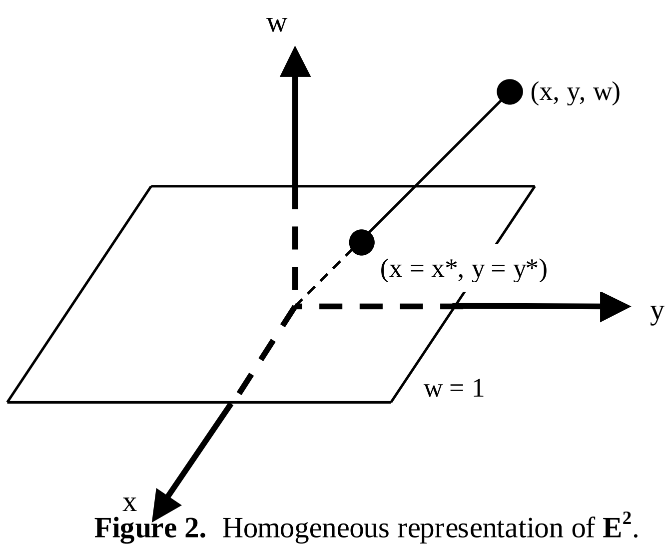


**Figure 2.** Homogeneous representation of $\mathbf{E}^2$.

Finally, a similar analysis for the case of $\mathbf{E}^3$ reveals that a point in $\mathbf{E}^3$ is the projection of a four-dimensional representation onto the hyper-plane described by w = 1 by the line that passes through the origin and through the point (x*, y*, z*, 1).[1]

[1] It is interesting to note that since we have now entered the realm of four-dimensional space, we can no longer illustrate the geometrical interpretation by a figure

Having investigated the geometrical interpretation of homogeneous coordinates for points in $\mathbf{E}^1$, $\mathbf{E}^2$, and $\mathbf{E}^3$, it can be seen that the underlying concept is that a space of a given dimension is represented by the equation w = 1 (a line, plane, or hyperplane) embedded in a space having a dimension one greater than that of the original space.

Using homogeneous coordinates, the transformations of points and vectors in a space of a given dimension are naturally performed in a space having a dimension one higher, and then this is succeeded by a projection back into the original space. This projection corresponds to dividing through by the transformed homogeneous component. The projective qualities of homogeneous coordinates will be explored later for perspective computation.

## Intersection of Two Lines

To become more familiar with homogeneous coordinates, we will now consider the problem of determining the intersection of two straight lines represented in homogeneous coordinates.

*Example 1:* In this example, the two lines are described by

$$\begin{aligned} x^* + 5\,y^* &= 10 \\ y^* &= 0 \end{aligned}$$

where (x*, y*) are the nonhomogeneous coordinates as in Figure 3.

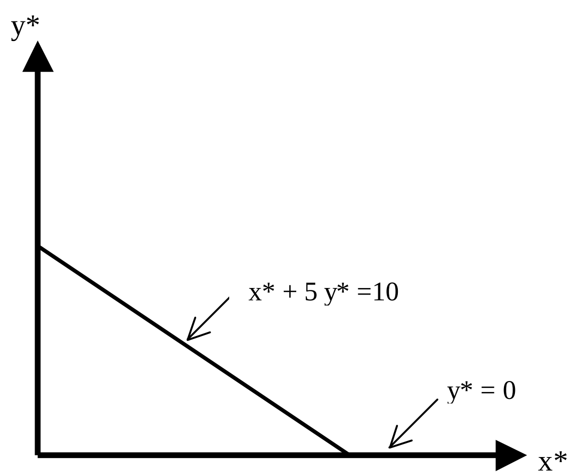


**Figure 3.** Intersection of two lines.

Rewriting using homogeneous coordinates yields

$$\begin{aligned} \frac{x}{w} + 5\,\frac{y}{w} - 10 &= 0 \\ \frac{y}{w} &= 0 \end{aligned}$$

Multiplying through by w,

$$\begin{aligned} x + 5y - 10w &= 0 \\ y \qquad\quad &= 0 \end{aligned}$$

Rewriting in matrix form,

$$\begin{bmatrix} 1 & 5 & -10 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since it will be necessary to invert the matrix to solve this equation, the matrix must be made square by adding a trivial equation. A suitable such equation is w = w. Adding this equation to the matrix form results in

$$\begin{bmatrix} 1 & 5 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} \qquad (1)$$

The inverse of the square matrix in the above equation is

$$\begin{bmatrix} 1 & -5 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pre-multiplying both sides of the matrix equation (1) by this inverse matrix results in

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 1 & -5 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$$

Multiplying out the right hand side,

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 10w \\ 0 \\ w \end{bmatrix}$$

Equating component-wise yields

$$x = 10\,w$$
$$y = 0$$

in homogeneous coordinates. Dividing each equation by w provides the nonhomogeneous coordinates

$$x^* = 10$$
$$y^* = 0$$

*Example 2:* Now, let us examine the result of applying the same procedure to determine the point of intersection of a pair of lines that are *parallel* as in Figure 4.

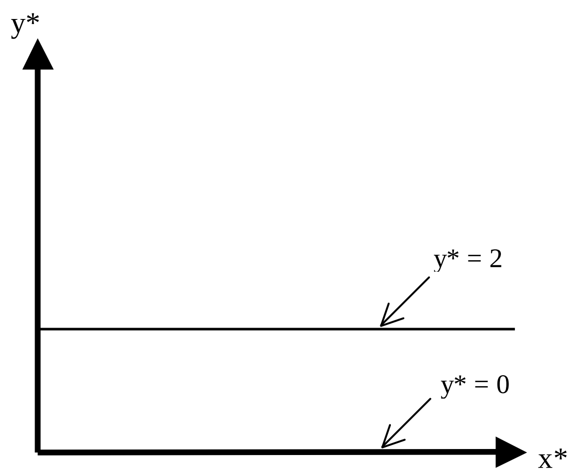


**Figure 4.** Intersection of parallel lines.

In this example, these lines are described by

$$y^* = 2$$
$$y^* = 0$$

in nonhomogeneous coordinates. Rewriting using homogeneous coordinates results in

$$\frac{y}{w} - 2 = 0$$
$$\frac{y}{w} = 0$$

Multiplying through by w

$$\begin{aligned} y - 2w &= 0 \\ y \quad\ \ &= 0 \end{aligned}$$

Rewriting in matrix form

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

Recall from the previous example that at this point we added the trivial equation w = w to form a square matrix. Let's see what happens if we try the same idea now.

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$$

However, note that the matrix thus formed has a column composed of zeros. Recall from linear algebra that this implies that the matrix is not invertible.

To solve the equation (2), we need to find a suitable equation to add, such that the resulting matrix is invertible. It can be seen that x = x is such an equation; this results in

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \tag{3}$$

The resulting matrix in the preceding equation has the following inverse:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Pre-multiplying both sides of equation (3) by this inverse matrix yields

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

Multiplying the righthand side yields

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

Now, $[x\ \ 0\ \ 0]^T$ is the point of intersection of the two lines; however, since the lines are parallel, this must be a representation of a point at infinity.

Let's consider this example in a more general setting. A pair of arbitrary parallel lines are given by

$$a\,y^* - b\,x^* = c$$
$$a\,y^* - b\,x^* = d$$

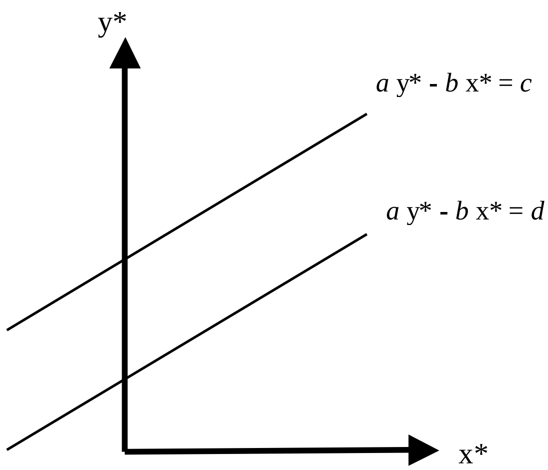


**Figure 5.** Intersection of arbitrary parallel lines.

(See Figure 5.) We will now compare the representation of their point of intersection (which again is actually a point at infinity).

Converting to homogeneous coordinates,

$$a\frac{y}{w} - b\frac{x}{w} - c = 0$$
$$a\frac{y}{w} - b\frac{x}{w} - d = 0$$

Multiplying through by w,

$$a\,y - b\,x - c\,w = 0$$
$$a\,y - b\,x - d\,w = 0$$

Converting to matrix form,

$$\begin{bmatrix} -b & a & -c \\ -b & a & -d \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we were again to try adding the trivial equation w = w , the resulting matrix would be

$$\begin{bmatrix} -b & a & -c \\ -b & a & -d \\ 0 & 0 & 1 \end{bmatrix}$$

However, note that the first two columns of the matrix are linearly dependent. Thus, the determinant of this matrix is zero, which again means that no inverse of this matrix exists.

Following the procedure of the last example, we can instead try adding the trivial equation x = x which yields

$$\begin{bmatrix} -b & a & -c \\ -b & a & -d \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

Now, this matrix is invertible; its inverse is

$$\frac{1}{a(c-d)} \begin{bmatrix} 0 & 0 & a(c-d) \\ -d & c & b(c-d) \\ -a & a & c \end{bmatrix}$$

Thus, we can solve for $\begin{bmatrix} x \\ y \\ w \end{bmatrix}$ yielding

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \frac{1}{a(c-d)} \begin{bmatrix} 0 & 0 & a(c-d) \\ -d & c & b(c-d) \\ -a & a & c \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

Multiplying out,

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \frac{1}{a(c-d)} \begin{bmatrix} a\,x(c-d) \\ b\,x(c-d) \\ 0 \end{bmatrix}$$

Provided that the parallel lines are distinct, that is, are not collinear, then c ≠ d, and this equation simplifies to

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \frac{x}{a} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

Since x/a is just a scale factor, we can conclude that the two-dimensional homogeneous vector $[a\ \ b\ \ 0]^T$ represents the point at infinity on an arbitrary line whose equation is of the form $a\,y^* - b\,x^* = k$. This provides a finite representation of infinite points and furthermore includes information about the direction in which the infinity “occurs”.

Another interpretation of w = 0 is that this represents a *vector* rather than a *point.* Note that this coordinate multiplies the translation elements of the transformation matrix. Thus, homogeneous coordinates having w = 0 are invariant under translation which is consistent with the fact that vectors are unaffected by translation.