Matrix Representation

- Linear transformation has a matrix representation
- Coordinates of a point as a vector

$$\begin{bmatrix} x & y \end{bmatrix} \text{ or } \begin{bmatrix} x \\ y \end{bmatrix} \text{ in 2D}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \text{ or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in 3D}$$

Row Vector Formulation

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} A \text{ in 2D}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{21}y & a_{12}x + a_{22}y \end{bmatrix}$$

$$x' = a_{11}x + a_{21}y$$

$$y' = a_{12}x + a_{22}y$$

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} A \text{ in 3D}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{21}y + a_{31}z & a_{12}x + a_{22}y + a_{32}z & a_{13}x + a_{23}y + a_{33}z \end{bmatrix}$$

$$x' = a_{11}x + a_{21}y + a_{31}z$$

$$y' = a_{12}x + a_{22}y + a_{32}z$$

$$z' = a_{13}x + a_{23}y + a_{33}z$$

Column Vector Formulation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix} \text{ in 2D, where } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{bmatrix}$$

$$x' = b_{11}x + b_{12}y$$

$$y' = b_{21}x + b_{22}y$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = B \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in 3D, where } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} b_{11}x + b_{12}y + b_{13}z \\ b_{21}x + b_{22}y + b_{23}z \\ b_{31}x + b_{32}y + b_{33}z \end{bmatrix} \Rightarrow \begin{matrix} x' = b_{11}x + b_{12}y + b_{13}z \\ y' = b_{21}x + b_{22}y + b_{23}z \\ z' = b_{31}x + b_{32}y + b_{33}z \end{matrix}$$

Correlation between matrices A & B

Note that

$$a_{ij} = b_{ji}$$

thus

$$A = B^t$$

Scaling

$$x' = s_x x$$
$$y' = s_y y$$

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$z' = s_z z$$

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

Rotation

- 2D

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned} \Rightarrow R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- 3D Rotation about X-axis (from pos. Y to pos. Z)

$$\begin{aligned} x' &= x \\ y' &= y\cos\theta - z\sin\theta \\ z' &= y\sin\theta + z\cos\theta \end{aligned} \Rightarrow R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation (cont'd)

- 3D Rotation about Y-axis (from pos. Z to pos. X)

$$\begin{aligned} x' &= x\cos\theta + z\sin\theta \\ y' &= y \\ z' &= -x\sin\theta + z\cos\theta \end{aligned} \Rightarrow R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

- 3D Rotation about Z-axis (from pos. X to pos. Y)

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned} \Rightarrow R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtleties

- Independent of "handedness" of coordinate system

Q: Which transformations are missing?
A: Translation

Q: Why?
A: Because it is nonlinear

Q: Why is it nonlinear?
A: Because it maps the origin to a non-zero point

Linear Transformations

$$f(\alpha x) = \alpha f(x)$$

$$f(x + y) = f(x) + f(y)$$

or, equivalently, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

if $\alpha = 0$,

$$f(0) = 0$$

That is, a linear transformation takes zero to zero

$$\mathbf{R}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\mathbf{S}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

but

$$\mathbf{T}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) \neq \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Translation

- Not a linear transformation, but an <u>affine</u> transformation, i.e. "linear plus translation"

Linear: $\begin{bmatrix}x'\\y'\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$

Translation: $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}t_x\\t_y\end{bmatrix}$

$$x' = x + t_x$$
$$y' = y + t_y$$

Translation (cont'd)

- Translation can be represented using a non-square matrix as:

$$\begin{bmatrix}x'\\y'\end{bmatrix} = A\begin{bmatrix}x\\y\\1\end{bmatrix}$$

$$A = \begin{bmatrix}1 & 0 & t_x\\0 & 1 & t_y\end{bmatrix}$$

Translation (cont'd)

- It is desirable to have square transformation matrices to be able to perform operations like matrix inversion and multiplication (for concatenation) and to have the representation of a point be the same before and after multiplication. So we modify the previous matrix by adding a row.

[0 0 1] for 2D

or

[0 0 0 1] for 3D

Translation (cont'd)

In 2D (3x3):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = T \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where $T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

In 3D (4x4):

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

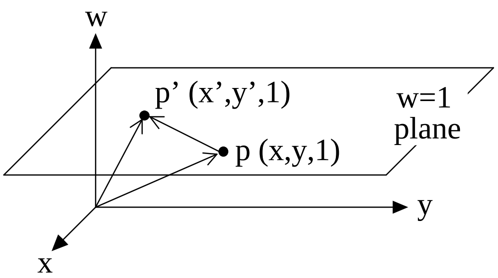
where $T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Translation (cont'd)

- Affine transformation is linear in the next dimension, i.e. an affine transformation in 2D is a linear transformation in 3D
- This can be seen simply by noting that the constant term becomes the extra coordinate in the next dimension
- That which is affine in one dimension is linear in the next dimension

Translation (concl'd)

- Consider in 2D:



$$x' = x + t_x w$$
$$y' = y + t_y w$$
$$w' = w = 1$$

- Translation in 2-space is like shear in 3-space
- Translation in 3-space is like shear in 4-space

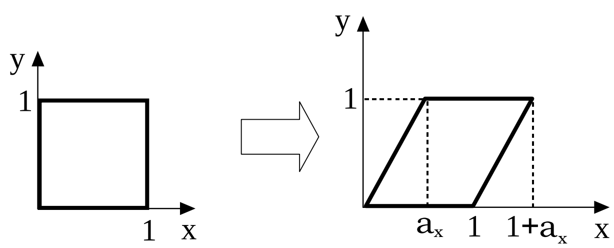
Shear Transformations

in 2D:

- Shear in the x direction (along y)

$$x' = x + a_x y$$
$$y' = y$$

$$SH_x = \begin{bmatrix} 1 & a_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



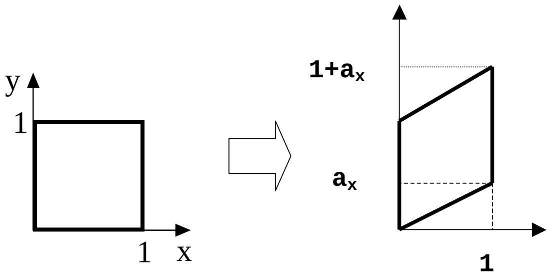
Shear Transformations

in 2D:

- Shear in the y direction (along x)

$$x' = x$$
$$y' = a_y x + y$$

$$SH_x = \begin{bmatrix} 1 & 0 & 0 \\ a_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Review of Transformations in 3D

- Scale

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotate

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Translate

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Review of Transformations in 3D

- Scale

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Translate

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Review of Transformations in 3D

- Rotate about the X axis

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotate about the Y axis

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotate about the Z axis

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$