Name:.. ID number:......................... Section:............

Midterm Exam

188 200 Discrete Mathematics and Linear Algebra

There are 7 parts; A-G, each for different topics. The total score is 120 points. Read the questions carefully then answer all of them. **All the additional information need for the exam are listed at the back of the exam.**

A: Logics

A.1 (8 points) Translate each English sentences (a – d) into propositions and each propositions (e – h) into English sentences (or Thai sentences).

Let $P(x)$: "x has a mobile phone."

Let $C(x, y)$: "x calls y."

a) Tom has a mobile phone, but never called Jo.

b) If Chon has a mobile phone then she will call somebody.

c) Everyone who has a mobile phone call somebody.

d) Everyone calls exactly one person.

e) $\exists x, \neg P(x)$

f) $\forall x, \neg P(x) \rightarrow \neg C(\text{Aump}, x)$

g) $\exists x \forall y, P(y) \rightarrow C(y, x)$

h) $\forall y \exists x, C(x, y) \rightarrow \neg C(y, x)$

A2. (4 points) Show that $(P \to Q) \to R$ and $P \to (Q \to R)$ are **not** equivalent by truth table.

A3. (4 points) Show that $(P \to (Q \to R)) \leftrightarrow ((P \wedge Q) \to R)$ is tautology using logical equivalences (provided at the back of this exam).

A3. (4 points) Validate the following argument by truth table

$$
\begin{array}{rl}
 & P \vee Q \\
 & P \to R \\
 & P \to \neg Q \\
\therefore & R
\end{array}
$$

.	.

A4. (4 points) Validate the following argument by rules of inference (provided at the end of this exam paper.)

$$
\begin{array}{rl}
 & \neg P \\
 & \neg Q \to R \\
 & R \to P \\
 & (Q \wedge \neg R) \to M \\
\therefore & M
\end{array}
$$

B. Methods of Proof

B1. (8 points) Prove that for all integers n, if $n \cdot (n-2)$ is odd then n is odd. (Hint use one of the following proofs: Direct proof, proof by contrapositive or proof by contradiction)

Proof: by...

Q.E.D.

B2. (8 points) Prove that for all integers n, if $3n + 2$ is even then n is even. **Do not use the same method of proof as B1.**

Proof: by...

Q.E.D.

C. Set Theory

C1. (4 points) Show that $A \cap B = A - (A - B)$ is true using set using set identities (provided at the end of this exam paper).

C2. (4 points) Prove that for all sets A, B and C, $[(A \cap B) - B] \cup [(A \cap C) - (C \cap B)] = [A \cap (B \cup C)] \cap \overline{B}$ using Venn diagram.

C3. (5 points) Circle True or False.

a) $\emptyset \cup \{a, b\} \ \in \ \{a, b, c, \emptyset\}$ True False

b) $\emptyset \in \{a, \{a, b\}, b, \{\ \}\}$ True False

c) $\{\emptyset\} \subseteq \{a, b, \{a\}, \{b\}, \{a, b\}, \{a, \emptyset\}, \{b, \emptyset\}, \{a, b, \emptyset\}\}$ True False

d) $\emptyset \subset \emptyset$ True False

e) If $A \cap \overline{B} \neq \emptyset$, then $|A \cap B| = 0$ True False

C4. (4 points) The barber puzzle: There is a town with a barber who shaves all the people who do not shave themselves. **Does the barber shave himself?** (Answer **Yes**, **No**, **Neither** or **Both**, then explain your answer.)

D. Sequence and Recurrence

D1. (4 points) Suppose the population of Thailand increase at a constant rate of 4% per year. The current population is 60 million. What would be the population of Thailand 25 years from now?

D2. (8 points) Consider the recurrence relation

$$a_k = 3 \cdot a_{k-1} + 4a_{k-2}$$

Find an explicit formula where $a_0 = 1$ and $a_1 = 3$ using distinct root formula theorem (provided at the back of this exam)

E. Mathematical Induction

E1. (8 points) Prove that $1 + 2 \cdot (2!) + 3 \cdot (3!) + \ldots + n \cdot (n!) = (n+1)! - 1$ for all integers $n \geq 1$ using mathematical induction. (fill in the appropriate answer)

Proof: by mathematical induction.

Let $P(n)$: ".."

Base case:

Induction hypothesis:

Inductive step:

Q.E.D.

E2. (8 points) Suppose that $c_1, c_2, c_3, \dots$ is a sequence defined as follows:

$$c_1 = 3, \quad c_2 = 5$$
$$c_k = c_{k-1} + 3c_{k-2} + 1 \text{ for all integers } k \geq 3.$$

Prove that c_n is odd for all integers $n \geq 3$ by strong mathematical induction. (Hint odd×odd = odd, odd + odd = even, odd - even = odd)

Proof: by strong mathematical induction.

Let $P(n)$: "..."

Base case:

Induction hypothesis:

Inductive step:

Q.E.D.

F. Basic of Counting, Permutation, Combination and Probability Theory

F1. (5 points) Circle True or False, or fill in your answers.

a) $P(n, r) = \binom{n}{r} \cdot n!$ True False

b) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ True False

c) $(n + r)^5 =$ ....

d) The permutation of the word "MISSISSIPPI" = ...

e) The 3-combination of a set $\{1, 2, 3, 4\}$ if repetition is allowed is $\binom{6}{3}$ True False

F2. (5 points) Suppose there is a group of 12 students consists of 7 men and 5 women.

a) How many ways can we form a team of five students that consists of three men and two women?

b) How many five-student teams contain at least two men?

c) How many five-student teams contain at most two men?

d) Suppose two students, one men and one women, always want to work together. How many five-student teams can be formed?

e) Suppose now that two students, one men and one women, do not want to work together. How many five-student teams can be formed?

F3. (4 points) Suppose a jar contains 10 blues balls, 8 reds balls and 12 green balls. We choose two balls at random, one after another without putting the balls back into the jar.

a) What is the probability that both balls are blue?

b) What is the probability that the second ball is red, but the first is not?

c) What is the probability that the second ball is red?

d) What is the probability that at least one of the ball chosen is blue?

F4. (3 points) Suppose there are two jars. The first jar contain 8 blues ball, 12 red balls and 10 green balls. The second jar contains 10 blues balls, 8 red balls and 12 green balls. We randomly choose a ball without knowing which jar the ball came from. If the ball is blue, what is the probability that it came from the first jar?

G. Functions and Relations

G1. (6 points) Circle True or False correct answer, or fill in your answer:

a) Let $A = \{1, 2, 3, 4, 5\}$.

Let $B = \{a, b, c, d\}$.

Let $f : A \to B$, $f(1) = b$, $f(2) = a$ $f(3) = d$ $f(4) = c$. $f(5) = a$.

f is one-to-one / onto / bijective / neither. (circle one)

b) Consider f from a). f does not have an inverse True False

c) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lceil x \rceil$ - $\lfloor x \rfloor$.

What is domain, the codomain and the range of f?

....

d) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$

Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$

What is $(g \circ f)(x)$ ....

e) Suppose $f : A \to B$ and $|A| = |B| = 100$.

If f is one-to-one then f is also onto. True False

f) How many cards must you pick to guarantee that you get every suites. ....

G2. (4 points) Which of the following are reflexive, symmetric, antisymmetric and/or transitive? Mark "**X**" if a relation **has** that property and mark "**0**" if a relation **does not** have that property.

a) $R_1 = \{(a, b) \mid a|b\}$

b) $R_2 = \{(a, b) \mid \text{if } a \geq 0 \text{ then } a = b \;\; \text{and} \;\; \text{if } a < 0 \text{ then } a = -b\}$

c) $R3 = \{(a, b) \mid a = \lceil b \rceil\}$

d) $R4 = \{(a, b) \mid a \geq b\}$

	Reflexive	Symmetric	Antisymmetric	Transitive
R_1				
R_2				
R_3				
R_4				

G3. (4 points) Let $R = \{(a, b) \mid |a| = |b|\}$. Is R an equivalent relation?

Logical Equivalences

Equivalences	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \wedge \mathbf{F} \equiv \mathbf{F}$ $p \vee \mathbf{T} \equiv \mathbf{T}$	Domination laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \wedge q \equiv q \wedge p$	Commutative law
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \to q \equiv \neg p \vee q$	Property of imply
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	Property of biconditional

Distinct Roots Theorem

Suppose a sequence $a_0, a_1, \ldots$ satisfies a recurrence relation in a form

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$$

for some real number A and B and all integers $k \geq 2$. If the characteristic equation

$$t^2 - At - B = 0$$

has two distinct root r and s, then $a_0, a_1, \ldots$ satisfies the explicit formula

$$a_n = C \cdot r^n + D \cdot s^n$$

where C and D are the numbers whose values are determineded by a_0 and a_1.

Rules of Inference

Addition
p
$\therefore p \vee q$

Conjunction
p
q
$\therefore p \wedge q$

Simplification
$p \wedge q$
$\therefore p$

Modus ponens
p
$p \rightarrow q$
$\therefore q$

Modus tollens
$\neg q$
$p \rightarrow q$
$\therefore \neg p$

Hypothetical syllogism
$p \rightarrow q$
$q \rightarrow r$
$\therefore p \rightarrow r$

Disjunctive syllogism
$p \vee q$
$\neg q$
$\therefore p$

Dilemma proof by division into cases
$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
$\therefore r$

Set Identities.

For all set A, B and C on the universal set U.

- Commutative laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Identity laws

$$A \cup \emptyset = A$$
$$A \cap U = A$$

- Complement laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

- Double complement law

$$\overline{(\overline{A})} = A$$

- Idempotent laws

$$A \cup A = A$$
$$A \cap A = A$$

- Universal bound laws

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

- De Morgan's laws

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

- Definition of set difference

$$A - B = A \cap \overline{B}$$