Lecture 5

188 200
Discrete Mathematics and Linear Algebra

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Overview of This Lecture

Propositional Logic

- Definition
- Proposition
- Logical Opertors
- Translating English Sentences to propositions
- Evalutate Truth Values
- Logical Equivalence/Tautology/Conditional Statement/ Biconditional Statement
- CNF/DNF
- Arguments and How to validate Arguments

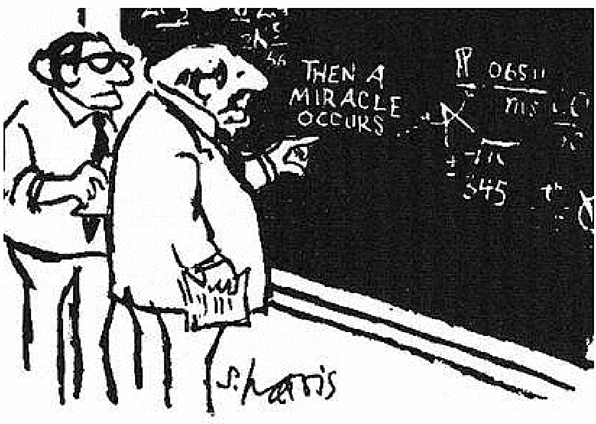
References

- Chapter 1: Section 2.1-2.3

Logics?

- The rules of logic specify the meaning of mathematical statement.
- Practical applications in many areas such as
  - design of computing machine
  - specification systems
  - AI
  - computer programming
  - ect.
- Will be a fundamental tool to prove a theorem – which is crucial in computer science.

Without Logic



"I think you should be more explicit here in step two."

Propositional Logic

Definition

Proposition (or statement) is a sentence that is true or false **but not both**.

Which one are propositions?

- $\sqrt{2} > 1$
- $1 + 1 = 1$
- Pig can fly.
- What time is it?
- $5^3 + \frac{7^2}{\sqrt{2}} + 3$
- $x + 1 = 2$
- $x + y > 0$

Notations

- Use small English letters to represent a proposition, e.g. p, q, r, s, . . .
  - Let p be "$1 + 1 = 1$"
  - Let q be "Pig can fly"

- The truth value of a proposition can be represented as "T" and "F" when
  - The truth value of p is T
  - The truth value of q is F

  it is true and false respectively.

Compound Propositions

Proposition can be atomic or compound.

Atomic Propositions

- Peter hates Lewis
- $\sqrt{2} > 1$
- p : where p represents a sentence "It is raining outside"

Compound Propositions are formed from existing propositions using **logical operators**, i.e. $\sim$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (imply), $\leftrightarrow$ (equivalent).

Example

- $p \wedge q$
- $\sim r$
- $\sim p \rightarrow (q \vee r)$

Logical Operators

- $\wedge$: and
- $\vee$: or
- $\sim$: not ($\neg$)
- $\rightarrow$: imply (if then)
- $\leftrightarrow$: if and only if (equivalent)
- $\oplus$: exclusive or

Truth Table is obtained by considering all possible combinations of truth values for propositions.

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
T	T	F	**T**	T	T	T	F
T	F	F	F	T	**F**	F	T
F	T	T	F	T	T	F	T
F	F	T	F	**F**	T	T	F

Translating from English Sentences to Propositions

Example 1:

1. "If it rains tomorrow then I will not go to school"
   - Let p be "It rains tomorrow"
   - Let q be "I go to school"
   - Hence $p \rightarrow \sim q$
2. "It is neither hot nor sunny"
   - Let h be "It is hot"
   - Let s be "It is sunny"
   - Hence $\sim h \wedge \sim s$
3. $0 < x \leq 3$
   - Let p be "$0 < x$"
   - Let q be "$x < 3$"
   - Let r be "$x = 3$"
   - Hence $p \wedge (q \vee r)$

Evaluate The Truth Value

Given a compound proposition, how do we compute the truth value?

Example 2: Evaluate the truth value of $(p \vee q) \rightarrow \sim (p \wedge q)$.

We will use truth table.

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$p \vee q \rightarrow \sim (p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	T

Logical Equivalences

Definition

The propositions p and q are called logically equivalent if they have identical truth values, denoted by $p \equiv q$.

- Using truth table.

Example 3: Show that $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Some Useful Logical Equivalences

Equivalences	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \wedge \mathbf{F} \equiv \mathbf{F}$ $p \vee \mathbf{T} \equiv \mathbf{T}$	Domination laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent laws
$\sim (\sim p) \equiv p$	Double negation law
$p \wedge q \equiv q \wedge p$	Commutative law
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$\sim (p \wedge q) \equiv \sim p \vee \sim q$ $\sim (p \vee q) \equiv \sim p \wedge \sim q$	De Morgan's laws

Evaluate Logical Equivalences

Example 4: Show that $\sim (p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

- We will use logical equivalences

$$
\begin{aligned}
\sim (p \vee (\sim p \wedge q)) &\equiv \sim p \wedge \sim (\sim p \wedge q) && \text{the second De Morgan's law} \\
&\equiv \sim p \wedge (\sim (\sim p) \vee \sim q) && \text{the first De Morgan's law} \\
&\equiv \sim p \wedge (p \vee \sim q) && \text{the double negation law} \\
&\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q) && \text{the distributive law} \\
&\equiv \mathbf{F} \vee (\sim p \wedge \sim q) && \text{since } p \wedge \sim p \equiv \mathbf{F} \\
&\equiv \sim p \wedge \sim q && \text{the identity law for } \mathbf{F}
\end{aligned}
$$

□

Tautology

Definition

A **tautology** (denoted by **t** or **T**) is a statement form that is always true regardless of the truth values of the individual statements.
A **contradiction** (denoted by **c** or **F**) is a statement form whose negation is a tautology (always false).

Some useful tautologies

- $p \vee \sim p$ — Law of excluded middle
- $p \rightarrow p$
- $p \leftrightarrow (p \wedge p)$
- $p \rightarrow q \leftrightarrow ((p \wedge \sim q) \rightarrow \mathbf{c})$ — reductio ad absurdum
- $p \rightarrow (p \vee q)$
- $(p \wedge q) \rightarrow p$
- $(p \wedge (p \rightarrow q)) \rightarrow q$ — modus ponens
- $((p \rightarrow q) \wedge \sim q) \rightarrow p$ — modus tollens

Conditional Statement

- **Conditional statement (or implication)**, $p \rightarrow q$, is only false when q is false and p is true.
- Related Implications
  - Contrapositive : $\sim q \rightarrow \sim p$
  - Converse : $q \rightarrow p$
  - Inverse : $\sim p \rightarrow \sim q$
- Caution ! : Only **the contrapositive form** is logically equivalent to conditional statement. The converse and the inverse are equivalent.
- **Note:** $p \rightarrow q$ is logically equivalent to $\sim p \vee q$.

Bi-conditional

The **biconditional** of p and q (denoted $p \leftrightarrow q$) is true if both p and q have the same truth values.

Variety of terminology

- p is necessary and sufficient for q
- if p then q, and conversely
- p if only if q
- p iff q

$p \leftrightarrow q$ is equivalent to $p \rightarrow q \wedge q \rightarrow p$.

Conjunctive Normal Form

Definition

A proposition is in **conjunctive normal form (CNF)** if it is a conjunction of clauses, where a clause is a disjunction of literals.

CNF will looks something like this

$$C_1 \wedge C_2 \wedge \ldots \wedge C_n$$

where C_i is a clause which is in the form

$$T_1 \vee T_2 \vee \ldots \vee T_m$$

Example:

- $p \vee q$
- $p \wedge q$
- $\sim p \wedge q$
- $p \wedge (q \vee r)$
- $(p \vee q) \wedge (p \vee \sim q)$
- $(p \vee q) \wedge (\sim r \vee s \vee t) \wedge u$

Disjunctive Normal Form

Definition

A formula (proposition) is in **disjunctive normal form (DNF)** if it is a disjunction of clauses, where a clause is a conjunctive of literals.

DNF will looks something like this

$$C_1 \vee C_2 \vee \ldots \vee C_n$$

where C_i is a clause which is in the form

$$T_1 \wedge T_2 \wedge \ldots \wedge T_m$$

Example:

- p
- $p \wedge q$
- $p \vee q$
- $\sim p \vee q \vee \sim r$
- $(p \wedge q) \vee r$
- $(p \wedge q \wedge \sim p) \vee (r \wedge \sim r)$

Conversion

Theorem

Every propositions can be converted into an equivalent proposition that is in **CNF**

Theorem

Every propositions can be converted into an equivalent proposition that is in **DNF**

Argument

- An **argument** is a sequence of propositions.
- Each proposition before the final one is called premise (or assumption or hypothesis).
- The final proposition is called the conclusion. the symbol $\therefore$ normally is placed before the conclusion.
- To say that an argument is **valid** means if all the promises are true then the conclusion must also be true.

Example 5:

If Ironman is a man, then Ironman is mortal

Ironman is a man

$\therefore$ Ironman is mortal

The above example has an abstract form as

$$
\begin{array}{rl}
 & p \rightarrow q \\
 & p \\
\therefore & q
\end{array}
$$

Identify premises and conclusion.

Solution:

$p \rightarrow q$ and p are premises, q is conclusion

Validating Argument

Using truth table!

1. Identify the premises and conclusion
2. Construct a **truth table** showing truth values of all the premises and the conclusion
3. A row in which all premises are true is called a critical row. If the conclusion of *every* critical row is true then the argument is **valid**, otherwise it is **invalid**.

Example 6: Validate the argument where premises are $p \vee (q \vee r)$ and $\sim r$, and the conclusion is $p \vee q$.

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\sim r$	$p \vee q$
T	T	T	T	T	F	T
T	T	F	T	T	T	**T**
T	F	T	T	T	F	T
T	F	F	F	T	T	**T**
F	T	T	T	T	F	T
F	T	F	T	T	T	**T**
F	F	T	T	T	F	F
F	F	F	F	F	T	F

The argument is valid

Example 7: Validate the argument where premises are $p \vee (q \vee r)$ and $\sim r$, and the conclusion is $(p \rightarrow q) \wedge \sim r$.

p	q	r	$q \vee r$	$p \rightarrow q$	$p \vee (q \vee r)$	$\sim r$	$(p \rightarrow q) \wedge \sim r$
T	T	T	T	T	T	F	F
T	T	F	T	T	T	T	**T**
T	F	T	T	F	T	F	F
T	F	F	F	F	T	T	**F**
F	T	T	T	T	T	F	F
F	T	F	T	T	T	T	**T**
F	F	T	T	T	T	F	F
F	F	F	F	T	F	T	T

The argument is invalid

Rule of Inferences

Although the truth table method always works, however, it is not convenient. Since the appropriate truth table must have 2^n lines where n is the number of atomic propositions.

Another way to show an argument is valid is to construct a formal proof. To do the formal proof we use rules of inference.

In rules of inference, premise(s) are written in a column and the conclusion is on the last line precede with the symbol $\therefore$ which denotes "therefore".

Example (Modus Ponens).

p	premises
$p \to q$	premises
$\therefore\ q$	since p is true and $p \to q$ is true q must be true

List of Rule of Inferences

Addition

p

$\therefore p \vee q$

Simplification

$p \wedge q$

$\therefore p$

Modus ponens

p

$p \rightarrow q$

$\therefore q$

Modus tollens

$\sim q$

$p \rightarrow q$

$\therefore \sim p$

Hypothetical syllogism

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Disjunctive syllogism

$p \vee q$

$\sim q$

$\therefore p$

Dilemma proof by division into cases

$p \vee q$

$p \rightarrow r$

$q \rightarrow r$

$\therefore r$

All these rules have been proved to be true!!

Validate Arguments with Rule or Inference

1. Identify the premise and conclusion
2. Write down arguments on a separate line. Usually starting with the premise(s). For each line, state clearly the reason.
3. Remember that argument written down is assumed or shown to be **true**!

Example 8: Given the following premises p, $p \rightarrow q$, $\sim q \vee r$ validate the conclusion r.

p	premise	(1)
$p \rightarrow q$	premise	(2)
q	Modus Ponens	(3)
$\sim q \vee r$	premise	(4)
$\sim (\sim q)$	from (3)	(5)
$\therefore\ r$	Disjunctive sylogism from (4) and (5)	

□