Lecture 8
188 200
Discrete Mathematics and Linear Algebra

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen University

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Overview

**Topic for today.**
- Combination
- Permutation with repeated elements
- Combination if repetition is allowed
- Interesting combinations
- Pascal’s formula

**Reference**: Section 6.4-6.6
Combination

Recall that in permutation, order is important.
▶ An arrangement of a bit string 1011 is different from 0111.

How do we count objects where order is not important?
▶ A set of bits \{1, 0, 1, 1\} is the same as \{0, 1, 1, 1\}.

Motivating example: How many ways can you make a set of two elements choosing from the set \{1, 2, 3, 4\}?

Solution: We can explicitly list all the sets:
\{0, 1\}, \{0, 2\}, \{0, 3\}
\{1, 2\}, \{1, 3\}
\{2, 3\}

This is call combinations.
If \( n \), and \( r \) are non-negative integers such that \( r \leq n \), an \( r \)-\emph{combination} is a subset of \( r \) of the \( n \) elements. The number of \( r \)-combination is

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

\( \binom{n}{r} \) is read “\( n \) choose \( r \)”

\( \binom{n}{r} \) can also be written as \( C(n, r) \).

The \( r \)-\emph{permutation} can be found by finding \( r \)-\emph{combination} and then permuting each \( r \)-\emph{combination} in each possible way.

\begin{itemize}
  \item Thus \( P(n, r) = \binom{n}{r} \cdot P(r, r) \).
    \begin{itemize}
      \item permuting each \( r \)-\emph{combination} = \( P(r, r) \).
    \end{itemize}
  \item This implies that \( \binom{n}{r} = \frac{P(n,r)}{r!} \) Note: \( P(r, r) = r! \).
\end{itemize}
Combination: Example

**Example:** How many ways can you choose a team of five students from a group of twelve students to work on a special project?

- In other words, what is a 5-combinations of 12?

**Solution:** The number of five-person team is

\[
\binom{12}{5} = \frac{12!}{5! \cdot (12 - 5)!} = \frac{12!}{5! \cdot 7!} = 11 \cdot 9 \cdot 8 = 792
\]
Combination : Example 2

**Example:** From the previous example, suppose two of the students want to work together. How many teams are there?
  ▶ Any team must have both student or neither.

**Solution:**
Combination : Example 3

**Example:** Still considering the choosing team example, suppose that two persons cannot work together. How many teams are there?

- Any team must have **either student or neither**.

**Solution:**

By sum rule.
Combination : Example 3 cont.

Solution:

Alternative solution.
Example: Suppose that the group of twelve students consists of five men and seven women.

1. How many five-person teams consist of three men and two women?

Solution:
2. How many five-person teams contain at least one man?

Solution:
Combination : Example 4 cont II.

3. How many five-person teams contain at most one man?

Solution:
Permutations of a Set with Repeated Elements

Suppose a set $S$ consist of $k$ types of object where:

- There are $n_1$ objects of type 1.
- There are $n_2$ objects of type 2.
- ... 
- There are $n_k$ objects of type $k$.

And suppose that $|S| = n_1 + n_2 + ... + n_k$. The permutation of $S$ is

$$
\frac{n!}{n_1! n_2! n_3! \ldots n_k!}
$$
Permutations of Set with Repeated Elements: Example

**Example:** What is a permutation of the word “MISSISSIPPI”?

**Solution:** There are four letters available.
- There are 4 of “S”.
- There are 4 of “I”.
- There are 2 of “P”.
- There are 1 of “M”.

There are 11 letters in total. Hence the permutation is

\[
\frac{11!}{4!4!2!1!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4!2!} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = 34650
\]
Combinations if Repetition is Allowed

The $r$-combination of a set of size $n$ when repetition is allowed:

$$\binom{r + n - 1}{r} = \frac{(r + n - 1)!}{r!(n - 1)!}.$$

Example: What is a 3-combinations of a set $\{1, 2, 3, 4\}$ if repetition is allowed?

- Elements can be used as many times as possible.

Solution: The combination is

$$\binom{3 + 4 - 1}{3} = \binom{6}{3} = \frac{6!}{3!3!} = \frac{6!}{3!3!} = 20.$$
Example: How many ways are there to select 5 bills from a cash box containing $1 bills, $2 bills, $5 bills, $10 bills, $20 bills, $50 bills, and $100 bills? Assume that the order in which bills are chosen does not matter and there are at least 5 bills of each type.

- There are 7 types of bills; $1, $2, $5, $10, $20, $50, $100.
- We draw 5 bills.
- The order that bills are drawn does not matter: Combination!
- There are at least 5 bills of each denominations: Repetition!
- In other words, what is a 5-combination of 7 if repetition is allowed?

Solution: The combination is

$$\binom{7 + 5 - 1}{5} = \binom{11}{5} = \frac{11!}{5!6!} = 462$$
Let’s try solving it without using the formula.

Note:

• The cash box has 7 compartments.
• These compartments are separated by 6 dividers. 
• Choosing 5 bills is the same as arranging 5 placeholders (denoted *) and 6 dividers (denoted | ).
Some Observations

Observation:

- **Arranging 5 ★ and 6 | is the same as choosing 5 “places” for the stars out of 11 total places.**

  ![Diagram](image)

- We can use our regular combination \( \binom{n}{r} \).
  - \( n \) is 11 and \( r \) is 5.
  
  \[
  \binom{11}{5} = \frac{11!}{5!6!} = 462
  \]

- This is how we derive the formula for combination with repetition.
Combination with Repetition: Example 3

**Example:** How many ways can we choose six cookies at a cookie shop that makes 4 types of cookie?

- In other words, what is 6-combination of a set of size 4 if repetition is allowed?

**Solution 1:** Reasoning with “stars” and “bars”

- Need six “stars” since we are choosing six cookies
- Need three “bars” to separate the cookies by type
  - e.g. choose 2 cookies of type-1, 2 of type-2, 1 of type-3, and 1 of type-4: 
    \[
    \text{* | * | * | * | * | * | * | * | *}
    \]
- Hence we are choosing 6 places out of 9 available places
- So, \( \binom{9}{6} = 84 \) ways to choose places to put stars.
Solution 2: Using the formula

- Since we choose six cookies, \( r = 6 \).
- Four possible cookies means \( n = 4 \).
- So, \( \binom{6+4-1}{6} = \binom{9}{6} = 84 \) ways to choose cookies!

Note: In exams, always use the formula, unless questions specify otherwise.
One More Example

**Example:** How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have if $x_1$, $x_2$, and $x_3$ are non-negative integers?

**Observation:** Solving this problem is the same as choosing 11 objects from a set of 3 objects such that $x_1$ objects of type one are chosen, $x_2$ objects of type two are chosen, and $x_3$ objects of type three are chosen.

**Solution :**

- $n = 3$
- $r = 11$
- So, there are $\binom{3+11-1}{11} \binom{13}{11} = 78$ ways to solve this equation.
## Formula Summary

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetition allowed</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$-permutation</td>
<td>No</td>
<td>$P(n, r) = \frac{n!}{(n-r)!}$</td>
</tr>
<tr>
<td>$r$-combination</td>
<td>No</td>
<td>$(\binom{n}{r}) = \frac{n!}{r!(n-r)!}$</td>
</tr>
<tr>
<td>$r$-permutation</td>
<td>Yes</td>
<td>$n^r$</td>
</tr>
<tr>
<td>$r$-combination</td>
<td>Yes</td>
<td>$(\binom{r+n-1}{r}) = \frac{(n+r-1)!}{r!(n-r)!}$</td>
</tr>
</tbody>
</table>
Some Interesting Combinations

1. \[ \binom{n}{n} = 1 \]

2. \[ \binom{n}{1} = n \]

3. \[ \binom{n}{0} = 1 \]

4. \[ \binom{n}{n-1} = n \]

5. \[ \binom{n}{r} = \binom{n}{n-r} \]
Pascal’s Formula

**Pascal’s formula:**

\[
\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}
\]

Using Pascal’s triangle, we can compute \( \binom{n}{r} \) recursively.

**Example:** Compute \( \binom{3}{2} \)

\[
\binom{3}{2} = \binom{2}{1} + \binom{2}{2}
\]

\[
= \binom{1}{0} + \binom{1}{1} + \binom{2}{2}
\]

\[
= 1 + 1 + 1
\]

\[
= 3
\]