Lecture 1

188 200
Discrete Mathematics and Linear Algebra

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Overview of This Lecture

- Course administration
- What is it about?
- What topics will be covered?
- Introduction to logic. We start with propositional logic.
  - Introduction to logic
  - Statement: atomic and compound statement
  - Logical connectives: $\wedge$, $\vee$ $\neg$, $\rightarrow$, $\leftrightarrow$, and their truth tables
  - Translating to and from English statements
  - Logical equivalences: De Morgan's Laws, Tautology
  - Conditional statement

References

- Chapter 1: Section 1.1 and 1.2

Course Administration

Lectures:

- Section 1: Wednesday and Friday @ 10:30 - 12:00
- Section 2: Tuesday and Thursday @ 10:30 - 12:00

There will be Q&A session in Thai once a week.

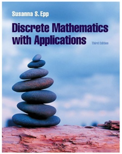
Lecturer: Dr.Pattarawit Polpinit

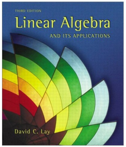
- Office: 4203 A
- Email: polpinit@kku.ac.th
- Website:
  http://gear.kku.ac.th/ polpinit/classes/188200/index.html
  (make sure you check the website regularly. All the updates will be posted there.)
- Office Hour: Tuesday and Thursday @ 2:00 - 3:00pm

Course Material

Textbook: There are two required textbooks, which should be available in both the university main library and the department library.

- **Discrete Mathematics:** Discrete Mathematics with Applications, S. Epp.
- **Linear Algebra:** Linear Algebra and Its Applications, D. Lay.





Slides: You are responsible for printing your own slides. They are available on the course website.

Any additional materials will be provided in class.

If You Need Help

If you have difficulties or problems on the course, you can do one of the following (in the order of preference):

- Come talk to me after class.
- Go to the Q&A session.
- Go to the office hour.
- Send me an email for question that doesn't need a lot of elaboration.
- Come see me at my office outside office hour: Please make sure you make an appointment by email first.

Do not hesitate to contact me if you have problem. Any problem is usually easier to fix if dealt with as soon as possible.

Lecture Conduct

Lecture are free-form, relaxed but serious.

- **Talking** is allowed only on course matters.
- **Attend lectures** be prepared for lecture.
- Try to **understand everything said in the lecture**. If you don't understand everything go read the textbooks or come to the Q&A or the office hour.
- **Ask question** in the lecture. Do not hesitate to stop me and ask question. What you don't understand is probably not understood by many other as well.
- **Stop the lecture** if you feel I'm going too fast. This is very important as it is the only way I know I'm going too fast or you need more time to digest.

Assessment

- **Attendance** 10 % : There will be random attendance checkings.
  - 2% will be deducted for each lecture you miss.
- **Quiz** 10 % : There will be two quizzes (5% each); one before midterm and the other before final.
- **Assignments** 10% : Assignments are usually problems from the end-of-the-chapter exercises in the textbook. You will be given two weeks for each assignment. They are due in class, any late submission will receive 10 % penalty per day on your grade on that assignment. You are expected to work individually on the assignments.
- **Midterm** 35% : The exam will include all discrete mathematics topics.
- **Final** 35 % : Only topics in linear algebra.

How to Succeed

- **Come to lecture** where you will see especially important examples and applications.
- The best way to obtain skills required to succeed in this course is to **do the assignments**. Discussion on the assignment is OK, but all the work handed in should be original, written by you on your own words.
- **Try to stay on schedule**. Each lecture builds on top of the previous one, so if you miss one lecture it will be difficult to catch up.
- **Read the book**. You should read the sections I note under the references in the overview of the lecture slide. You will get additional informations and clarification on some of the thing that you are still unclear from the lectures.

What is 188 200 is about?

Discrete Mathematics and **Linear Algebra**

- **Discrete mathematics** is the study of mathematical structures that are fundamentally discrete rather than continuous. Its concepts and notations are useful in studying and describing objects and problems in computer algorithms and programming languages, and have applications in cryptography, automated theorem proving, and software development.
- **Linear algebra** studies matrix algebra, determinants, vectors, linear spaces, linear transformations, and systems of linear equations. Linear algebra is one of the most fundamental concepts in modern mathematics.

Goals of 188 200

Main goal is to introduce students to a variety of mathematical tools that are keys in computer engineering.

- Learn how to write statement rigorously.
- Learn how to read/write theorem, lemma etc.
- Learn how to write rigorous proofs.
- Become proficient in the language of linear algebra, to be able to use it in theoretical aspect and applications from computer engineering and other disciplines.

Topics covered in 188 200 I

Discrete Mathematics

- Logic
  - Propositional Logic
  - Predicate Logic
- Method of Proof
  - Direct Proof
  - Indirect Proof
  - and more proofs
- Set Theory
  - Set Properties
  - Set Operations
  - Russell's Paradox and the Halting problem
- Sequences and Summations
  - Geometric and Arithmetic Sequence
  - Summation Properties
- Mathematical Induction

Topics covered in 188 200 II

    - Basic of Induction
    - Strong Induction
- Counting
    - Basic of Counting
    - Permutation and Combinations
    - Probability Theory
- Functions
    - One-to-one and Onto Functions
    - Inverse Functions and Compositions of Functions

Linear Algebra

- Systems of linear equations
    - Introduction to System of Linear Equation
    - Row Reduction and Echelon Forms
- Vectors and Matrix Equations
    - Vectors
    - The Solution Set of Linear Systems

Topics covered in 188 200 III

  - Linear Independence
  - Linear Transformation
- Matrix algebra.
  - Matrix operations
- Determinants
  - Properties of Determinants
- Vector Spaces
  - Vector Spaces
  - Subspace
- Eigenvalues and Eigenvectors

What is Logic?

Logics are formal languages for formalizing reasoning, in particular for representing information such that conclusion can be drawn.

A logic involves

- A language with a syntax for specifying what is a legal expression in the language; syntax defines well formed sentences in the language
- Semantics for associating elements of the language with elements of some subject matter. Semantics defines the meaning of sentences (link to the world); i.e., semantics defines the truth of a sentence with respect to each possible world
- Inference rules for manipulating sentences in the language

Original motivation: Early Greeks, settle arguments based on purely rigorous (symbolic/syntactic) reasoning starting from a given set of premises.

Propositional Logic

Proposition (or a statement) is a sentence that is true or false but not both.

Which one are propositions?

- $\sqrt{2} > 1$
- $1 + 1 = 1$
- Pig can fly.
- What time is it?
- $5^3 + \frac{7^2}{\sqrt{2}} + 3$
- Read this sentence carefully.
- $x + 1 = 2$
- $x + y > 0$

Compound Propositions

Proposition can be atomic or compound.

Atomic Propositions

- Peter hates Lewis $\longrightarrow$ p
- It is raining outside $\longrightarrow$ q
- $\sqrt{2} > 1$ $\longrightarrow$ r

Compound Propositions (or statement form or propositional form) are formed from existing propositions using logical operators, i.e. $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (imply), $\leftrightarrow$ (equivalent).

Example

- Peter hates Lewis and it is raining outside. : $p \wedge q$
- $\sqrt{2} \not> 1$: $\neg r$
- If Peter doesn't hate Lewis then It is raining or $\sqrt{2} > 1$: $\neg p \rightarrow (q \vee r)$

Logical Operator

- $\wedge$: and
- $\vee$: or
- $\neg$: not
- $\rightarrow$: imply (if then)
- $\leftrightarrow$: if and only if (equivalent)

Truth Table : is obtained by considering all possible combinations of truth values for propositions.

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Well-Formed Formulas

Propositions and compund propositions must follow syntax.
Statements that follows syntax are called well-formed fomulas.

Atomic propositions are wffs.

- p, q
- Apple is red.

Compound propositions that are wffs.

- $p \wedge q$
- $q \rightarrow r$
- $(p \vee q) \leftrightarrow r$

What about one of these?

- $p\neg\neg$
- $q \rightarrow\rightarrow r$
- $\neg\neg\neg p$
- $p \rightarrow q \vee p \rightarrow q$

Evaluate The Truth Value

Given a compound proposition, how do we compute the truth value?

Example:

Evaluate the truth value of $p \vee q \rightarrow \neg(p \wedge q)$.

We will use truth table.

p	q	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q \rightarrow \neg(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	T

Translating English Sentence

- English (even Thai and every other human languages) is ambiguous.

Example "If you are under 4 feet tall and you are younger than 16 years old then you cannot ride the roller coaster."

- You can ride the roller coaster : p
- You are under 4 feet tall : q
- You are younger than 16 years old : r
- $(q \wedge r) \rightarrow \neg p$

Caution!

- Jim is tall and Jim is thin
- What is the negation?
- Is it "Jim is not tall and Jim is not thin."?
- It actually is "Jim is not tall or Jim is not thin."
- This is called De Morgan's Laws.

Logical Equivalences

Definition : The propositions p and q are called logically equivalent if they have identical truth values, denoted by $p \equiv q$.

- One way to determine if two propositions are logically equivalent is to use truth table.

Example Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

- The truth table can also be used to show that two propositions are not equivalent.

Some Useful Logical Equivalences

Equivalences	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \wedge \mathbf{F} \equiv \mathbf{F}$ $p \vee \mathbf{T} \equiv \mathbf{T}$	Domination laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \wedge q \equiv q \wedge p$	Commutative law
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws

Evaluate Logical Equivalences

Example

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

- We will use logical equivalences

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{from the second De Morgan's law} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{from the first De Morgan's law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{from the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{from the distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{since } p \wedge \neg p \equiv \mathbf{F} \\
&\equiv \neg p \wedge \neg q && \text{from the identity law for } \mathbf{F}
\end{aligned}
$$

Q.E.D.

Tautology

Definition : A tautology (denoted by $\mathbf{t}$) is a statement form that is always true regardless of the truth values of the individual statements.

A contradiction (denoted by $\mathbf{c}$) is a statement form whose negation is a tautology.

Some useful tautologies

- $p \vee \neg p$ Law of excluded middle
- $p \rightarrow p$
- $p \leftrightarrow (p \wedge p)$
- $p \rightarrow q \leftrightarrow ((p \wedge \neg q) \rightarrow \mathbf{c})$ reductio ad absurdum
- $p \rightarrow (p \vee q)$
- $(p \wedge q) \rightarrow p$
- $(p \wedge (p \rightarrow q)) \rightarrow q$ modus ponens
- $((p \rightarrow q) \wedge \neg q) \rightarrow p$ modus tollens

Conditional Statement

- **Conditional statement (or implication)**, $p \rightarrow q$, is only false when q is false and p is true.
- Related Implications
  - Contrapositive : $\neg q \rightarrow \neg p$
  - Converse : $q \rightarrow p$
  - Inverse : $\neg p \rightarrow \neg q$
- Caution ! : Only the contrapositive form is logically equivalent to conditional statement. The converse and the inverse are equivalent.
- **Note:** $p \rightarrow q$ is logically equivalent to $\neg p \vee q$.

Conditional Statement II

- Conditional statement plays an important role in reasoning a variety of terminology used to refer to implication ($p \rightarrow q$).

- if p then q
- if p, q
- p is sufficient for q
- q if p
- q when p
- a necessary condition for p is q
- p implies q
- p only if q
- a sufficient condition for q is p
- q whenever p
- q is necessary for p
- q follows from p

Note : the mathematical implication of implication is independent of a cause and effect relationship between hypothesis (p) and conclusion (q), this is normally present when we use implication in language. e.g. $p \rightarrow q$ where p is $1 + 1 = 2$ and q is "it is raining today".

Bi-conditional

The **biconditional** of p and q (denoted $p \leftrightarrow q$) is true if both p and q have the same truth values.

Variety of terminology

- p is necessary and sufficient for q
- if p then q, and conversely
- p if only if q
- p iff q

$p \leftrightarrow q$ is equivalent to $p \rightarrow q \wedge q \rightarrow p$.

Argument

An **argument** is a sequence of propositions. All the propositions before the final one are called premises (or assumption or hypothesis). The final proposition is called the conclusion.

An argument is valid whenever the truth of all its premises implies the truth of its conclusion.

How to show q logically follows from the premises $p_1 \wedge p_2 \wedge \ldots \wedge p_n$?

Basically, we show that the argument $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is tautology.

Valid Argument

Example Show that the argument where premises are $p \vee (q \vee r)$ and $\neg r$, and the conclusion is $p \vee q$, is valid.

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\neg r$	$p \vee q$
T	T	T	T	T	F	T
T	T	F	T	T	T	T
T	F	T	T	T	F	T
T	F	F	F	T	T	T
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	F	F	T	F

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T	T	T	T	T	F	T
T	T	F	T	T	T	T
T	F	T	T	T	F	T
T	F	F	F	T	T	T
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	F	F	T	F

Valid Argument

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p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\neg r$	$p \vee q$
T	T	T	T	T	F	T
T	T	F	T	T	T	**T**
T	F	T	T	T	F	T
T	F	F	F	T	T	**T**
F	T	T	T	T	F	T
F	T	F	T	T	T	**T**
F	F	T	T	T	F	F
F	F	F	F	F	T	F

The argument is valid

Rule of Inferences

Although the truth table method always works, however, it is not convenient. Since the appropriate truth table must have 2^n lines where n is the number of atomic propositions.

Another way to show an argument is valid is to construct a formal proof. To do the formal proof we use rules of inference. The rules of inference is a function form sets of wffs to wffs. In rules of inference, premise(s) are written in a column and the conclusion is on the last line precede with the symbol $\therefore$ which denotes "therefore". For example the following rule is called Modus Ponens.

$$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \therefore q \end{array}$$

In words, if p and $p \rightarrow q$ are true q is true.

Rule of Inferences Example

Example

If you study the 188 200 materials then you will pass the exam. ($p \to q$)

you study the 188 200 material. (p)

$\therefore$ You will pass. (q)

Nothing deep here, but remember formal reason is that $((p \to q) \land p) \to q$ is tautology.

List of Rule of Inferences

Addition

p

$\therefore p \vee q$

Simplification

$p \wedge q$

$\therefore p$

Modus ponens

p

$p \rightarrow q$

$\therefore q$

Modus tollens

$\neg q$

$p \rightarrow q$

$\therefore \neg p$

Hypothetical syllogism

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Disjunctive syllogism

$p \vee q$

$\neg q$

$\therefore p$

Dilemma proof by division into cases

$p \vee q$

$p \rightarrow r$

$q \rightarrow r$

$\therefore r$

Recap

- Introduction to logic
- Propositions (statement)
- Logical Operators
- wffs
- Truth table
- Logical equivalences
- Tautologies
- Conditional statements
- Bi-conditional statemenets
- Valid argument
- Rule of inferences