Overview

Topics for today.

▶ A transformation
▶ Matrix transformation
▶ Check weather a transform is linear
▶ The matrix of a linear transformation
▶ Function properties of linear transformation

Reference: Section 1.8-1.9
A Transformation

\[ A \overrightarrow{x} = \overrightarrow{b} \quad \text{VS} \quad x_1 \overrightarrow{a_1} + \ldots + x_n \overrightarrow{a_n} = \overrightarrow{b} \]

- They are basically the same.
- \( A \overrightarrow{x} = \overrightarrow{b} \) is more useful, e.g. for computer graphic and signal processing.
- You can think of \( A \) as a tool to transform \( \overrightarrow{x} \) to \( \overrightarrow{b} \).

Example:

\[
\begin{bmatrix}
2 & -4 \\
3 & -6 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
3
\end{bmatrix}
= 
\begin{bmatrix}
-8 \\
-12 \\
-4
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & -4 \\
3 & -6 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

These show that the multiplication by \( A \) transform \( \overrightarrow{x} \) into \( \overrightarrow{b} \) and \( \overrightarrow{u} \) into \( \overrightarrow{0} \) respectively.
With this new perspective, solving $A\vec{x} = \vec{b}$ is to find all $\vec{x}$ in $\mathbb{R}^2$ that can be transformed to $\vec{b}$ in $\mathbb{R}^3$ through multiplication by $A$.

**Definition:** A transformation (or mapping) $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that is an assignment to each $\vec{x}$ in $\mathbb{R}^n$ $T(\vec{x})$ in $\mathbb{R}^m$. $\mathbb{R}^n$ is the **domain** of $T$ and $\mathbb{R}^m$ is the **codomain** of $T$. For each $\vec{x}$, $T(\vec{x})$ is an **image** of $\vec{x}$ and the set of all images is the **range** of $T$. 
Matrix Transformations

- Focus on mapping with matrix multiplication $A$.
- Sometimes denote a matrix transformation by $\vec{x} \mapsto A\vec{x}$.

**Example 1:** Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$.

Then if $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$
Example 1, cont.

\[ T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \]

Domain : \( \mathbb{R}^2 \)

Codomain : \( \mathbb{R}^3 \)
Example 2: Let \( A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}, \overrightarrow{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \)

\( \overrightarrow{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix} \) and \( \overrightarrow{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \) Define a transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by \( T(\overrightarrow{x}) = A(\overrightarrow{x}). \)

1. Find an \( \overrightarrow{x} \) in \( \mathbb{R}^3 \) whose image under \( T \) is \( \overrightarrow{b} \).
2. Is there more than one \( \overrightarrow{x} \) under \( T \) whose image is \( \overrightarrow{b} \) (uniqueness problem)
3. Determine \( \overrightarrow{c} \) is in the range of the transformation \( T \). (existence problem)

Solution:
Example 2, cont.

**Solution:** (cont.)
Is a Transformation Linear?

**Definition:** A transformation \( T \) is **linear** if:

1. \( T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \) for all \( \vec{u}, \vec{v} \) in the domain of \( T \).
2. \( T(c \vec{u}) = cT(\vec{u}) \) for all \( \vec{u} \) and all scalars \( c \).

**Fact:** Every matrix transformation is a linear transformation.

**Corollary:**

3. If \( T(\vec{0}) = \vec{0} \)
4. \( T(c \vec{u} + d \vec{v}) + cT(\vec{u}) + dT(\vec{v}) \)
   for all \( \vec{u}, \vec{v} \) in the domain of \( T \) and all scalars \( c \) and \( d \).

(4) can be generalized to (known as **superposition principle**)

\[
T(c_1 \vec{v}_1 + \ldots + c_p \vec{v}_p) = c_1 T(\vec{v}_1) + \ldots + c_p T(\vec{v}_p)
\]
Example 3: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r \vec{x}$ where $r$ is a scalar. Show that $T$ is a linear transformation.

Note: $T$ is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$.

Solution:
Example 4: Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation which maps $\vec{e}_1$ into $\vec{y}_1$ and $\vec{e}_2$ into $\vec{y}_2$. Find the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution:
Example 4, cont.

Solution to Example 4 (cont.)
Example 5

Example 5: Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}.$$ Show that $T$ is not a linear transformation.

Solution:
The Matrix of a Linear Transformation

► When talk about $T$, usually want to find a “formula” for $T(\overrightarrow{x})$.

► This can be done by observing what $T$ does to the columns of the $n \times n$ identity matrix.

**Definition:** An **identity matrix** (denoted as $I_n$) is an $n \times n$ matrix with 1's on the left-to-right diagonal and 0's elsewhere. The $i$-th column of $I_n$ is labeled $\overrightarrow{e_i}$.

**Example 6:** Suppose that $T$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\overrightarrow{e_1}) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \quad \text{and} \quad T(\overrightarrow{e_2}) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$ 

Compute $T(\overrightarrow{x})$ for any $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. 
Example 6, Solution

Solution:
**Standard Matrix**

**Theorem 10:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $A$ such that

$$T(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n$$

In fact, $A$ is the $m \times n$ matrix whose $j$-th column is the vector $T(\vec{e}_j)$ is the $j$-th column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

**Note:** $A$ is called standard matrix for the linear transformation $T$. 
Example 7

Example 7: Let $A$ be a $3 \times 2$ matrix. Find $A$ from the following equation:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

Solution:
Rotation Transformation

**Example 8:** Find the standard matrix \( A \) of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which rotates a point about the origin through an angle of \( \varphi \) with counterclockwise rotation for a positive angle.

**Solution:**

![Diagram of rotation transformation](image)
Example 8, cont.

Solution to Example 8: (cont.)
Example 9 Find the standard matrix of the following transformation, if the input is the unit square.

Reflection in the $x_2$-axis

<table>
<thead>
<tr>
<th>Image of the Unit Square</th>
<th>Standard Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0] \begin{bmatrix} 0 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 \ 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Note: For more standard matrices check Table 1-4 in section 1.9.
Function Properties of Linear Transformation

Definition: \( T : \mathbb{R}^n \to \mathbb{R}^m \) is said to be onto \( \mathbb{R}^m \) if each \( \vec{b} \) in \( \mathbb{R}^m \) is the image of at least one \( \vec{x} \) in \( \mathbb{R}^n \).

- Equivalently, \( T \) is onto if the range of \( T = \) the codomain \( \mathbb{R}^m \).
- This is an existence question, since \( T \) is not onto if \( \exists \vec{b} \) for which \( T(X) = \vec{b} \) has no solution.

Definition: \( T : \mathbb{R}^n \to \mathbb{R}^m \) is said to be one-to-one if each \( \vec{b} \) in \( \mathbb{R}^m \) is the image of at most one \( \vec{x} \) in \( \mathbb{R}^n \).

- Equivalently, \( T \) is one-to-one if \( \forall \vec{b} \) either has a unique solution or none at all.
- This is a uniqueness question, since \( T \) is not one-to-one if \( \exists \vec{b} \) that is the image of more than one \( \vec{x} \) in \( T \).
Determine if Transformation is Onto/One-to-one

**Theorem 11:** Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation. Then \( T \) is one-to-one iff \( T(\vec{x}) = \vec{0} \) has only the trivial solution.

**Theorem 12:** Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation and \( A \) be the standard matrix for \( T \). Then

a. \( T \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^m \) iff the columns of \( A \) span \( \mathbb{R}^m \).

b. \( T \) is one-to-one iff the columns of \( A \) are linearly independent.
Example 10: Let $A$ be the linear transformation whose standard matrix is

$$
A = \begin{bmatrix}
1 & -4 & 8 & 1 \\
0 & 2 & -1 & 3
\end{bmatrix}
$$

Does $T$ maps $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is $T$ a one-to-one mapping?

Solution:

- $A$ is in echelon form that has pivot in every rows.
  - This implies that for each $\vec{b}$ in $\mathbb{R}^3$, $A\vec{x} = \vec{b}$ has a solution.
  - By Theorem 4 in Section 1.4, this implies that $T$ maps $\mathbb{R}^4$ onto $\mathbb{R}^3$.

- Since $A$ has a free variable, each $\vec{b}$ is the image of more than one $\vec{x}$.
  - $T$ is not one-to-one.
Example 11: Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$.
Show that $T$ is a one-to-one linear transformation. Does $T$ maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.

Solution: Given that

$$T(\overrightarrow{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The columns of standard matrix $A$ are linearly independent since they are not multiple of each other.
  - By Theorem 12b, $T$ is one-to-one.
- $T$ maps onto $\mathbb{R}^3$ if the columns of $A$ span $\mathbb{R}^3$.
  - Which is true iff $A$ has a pivot in every rows. This is impossible, hence $T$ is not onto $\mathbb{R}^3$. 
Recap

- A transformation
- Matrix transformation
- Check weather a transform is linear
- The matrix of a linear transformation
- Function properties of linear transformation

Next time, we will start Chapter 2, matrix algebra.