Lecture 8

188 200
Discrete Mathematics and Linear Algebra

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Overview

Topic for today.

- Combination
- Permutation with repeated elements
- Combination if repetition is allowed
- Interesting combinations
- Pascal's formula

Reference : Section 6.4-6.6

Combination

Recall that in permutation, order is important.

- An arrangement of a bit string 1011 is different from 0111.

How do we count objects where order is not important?

- A set of bits {1, 0, 1, 1} is the same as {0, 1, 1, 1}.

Motivating example: How many ways can you make a set of two elements choosing from the set $\{1, 2, 3, 4\}$?

Solution: We can explicitly list all the sets:

$\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$

$\{1, 2\}$, $\{1, 3\}$

$\{2, 3\}$

This is call **combinations**.

Combination 2

If n, and r are non-negative integers such that $r \leq n$, an **r-combination** is a subset of r of the n elements. The number of r-combination is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- $\binom{n}{r}$ is read "n choose r"
- $\binom{n}{r}$ can also be written as $C(n, r)$.
- The r-permutation can be found by finding r-combination and then permuting each r-conbination in each possible way.
  - Thus $P(n, r) = \binom{n}{r} \cdot P(r, r)$.
    - permuting each r-combination $= P(r, r)$.
  - This implies that $\binom{n}{r} = \frac{P(n,r)}{r!}$ Note: $P(r, r) = r!$.

Combination : Example

Example: How many ways can you choose a team of five students from a group of twelve students to work on a special project?

- In other words, what is a 5-combinations of 12?

Solution: The number of five-person team is

$$\binom{12}{5} = \frac{12!}{5! \cdot (12-5)!} = \frac{12!}{5! \cdot 7!} = 11 \cdot 9 \cdot 8 = 792$$

Combination : Example 2

Example: From the previous example, suppose two of the students want to work together. How many teams are there?

- Any team must have both student or neither.

Solution:

Combination : Example 3

Example: Still considering the choosing team example, suppose that two persons cannot work together. How many teams are there?

- Any team must have either student or neither.

Solution:

By sum rule.

Combination : Example 3 cont.

Solution:

Alternative solution.

Combination : Example 4

Example: Suppose that the group of twelve students consists of five men and seven women.

1. How many five-person teams consist of three men and two women?

Solution:

Combination : Example 4 cont.

2. How many five-person teams contain at least one man?

Solution:

Combination : Example 4 cont II.

3. How many five-person teams contain at most one man?

Solution:

Permutations of a Set with Repeated Elements

Suppose a set S consist of k types of object where:

- There are n_1 objects of type 1.
- There are n_2 objects of type 2.
- ...
- There are n_k objects of type k.

And suppose that $|S| = n_1 + n_2 + \ldots + n_k$. **The permutation of S is**

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!n_2!n_3!\ldots n_k!}$$

Permutations of Set with Repeated Elements: Example

Example: What is a permutation of the word "MISSISSIPPI"?

Solution: There are four letters available.

- There are 4 of "S".
- There are 4 of "I".
- There are 2 of "P".
- There are 1 of "M".

There are 11 letters in total. Hence the permutation is

$$\frac{11!}{4!4!2!1!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4!2!}$$
$$= 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5$$
$$= 34650$$

Combinations if Repetition is Allowed

The r-combination of a set of size n when repetition is allowed:

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!(n-1)!}.$$

Example: What is a 3-combinations of a set $\{1,2,3,4\}$ if repetition is allowed?

- Elements can be used as many times as possible.

Solution: The combination is

$$\binom{3+4-1}{3} = \binom{6}{3} = \frac{6!}{3!3!} = 20$$

Combinations with Repetition: Example 2

Example: How many ways are there to select 5 bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which bills are chosen does not matter and there are at least 5 bills of each type.

- There are 7 types of bills; \$1, \$2, \$5, \$10, \$20, \$50, \$100.
- We draw 5 bills.
- The order that bills are drawn does not matter: Combination!
- There are at least 5 bills of each denominations: Repetition!
- In other words, what is a 5-combination of 7 if repetition is allowed?

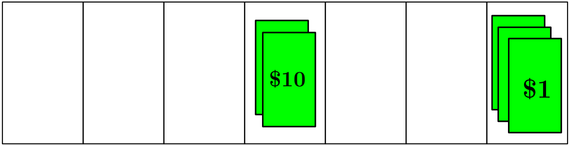
Solution: The combination is

$$\binom{7+5-1}{5} = \binom{11}{5} = \frac{11!}{5!6!} = 462$$

Let's try solving it without using the formula.

Note:

- The cash box has 7 compartments.
- These compartments are separated by 6 dividers
- Choosing 5 bills is the same as arranging 5 placeholders (denoted *) and 6 dividers (denoted |).



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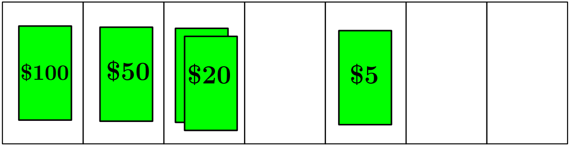

$$\Longrightarrow *|*|**||*||$$

Some Observations

Observation :

- **Arranging 5 $\ast$ and 6 $\vert$ is the same as choosing 5 "places" for the stars out of 11 total places.**

$\ast$	$\vert$	$\vert$	$\ast$	$\ast$	$\vert$	$\vert$	$\ast$	$\vert$	$\ast$	$\vert$

- We can use our regular combination $\binom{n}{r}$.
- n is 11 and r is 5.

$$\binom{11}{5} = \frac{11!}{5!6!} = 462$$

- This is how we derive the formula for combination with repetition.

Combination with Repetition: Example 3

Example: How many ways can we choose six cookies at a cookie shop that makes 4 types of cookie?

• In other words, what is 6-combination of a set of size 4 if repetition is allowed?

Solution 1 : Reasoning with "stars" and "bars"

- Need six "stars" since we are choosing six cookies
- Need three "bars" to separate the cookies by type
  - e.g. choose 2 cookies of type-1, 2 of type-2, 1 of type-3, and 1 of type-4: $**|**|*|*$
- Hence we are choosing 6 places out of 9 available places
- So, $\binom{9}{6} = 84$ ways to choose places to put stars.

Combination with Repetition: Example 3 cont.

Solution 2 : Using the formula

- Since we choose six cookies, $r = 6$.
- Four possible cookies means $n = 4$.
- So, $\binom{6+4-1}{6} = \binom{9}{6} = 84$ ways to choose cookies!

Note: In exams, always use the formula, unless questions specify otherwise.

One More Example

Example: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have if x_1, x_2, and x_3 are non-negative integers?

Observation: Solving this problem is the same as choosing 11 objects from a set of 3 objects such that x_1 objects of type one are chosen, x_2 objects of type two are chosen, and x_3 objects of type three are chosen.

Solution :

- $n = 3$
- $r = 11$
- So, there are $\binom{3+11-1}{11}\binom{13}{11} = 78$ ways to solve this equation.

Formula Summary

Type	Repetition allowed	Formula
r-permutation	No	$P(n, r) = \frac{n!}{(n-r)!}$
r-combination	No	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$
r-permutation	Yes	n^r
r-combination	Yes	$\binom{r+n-1}{r} = \frac{(n+r-1)!}{r!(n-r)!}$

Some Interesting Combinations

1. $$\binom{n}{n} = 1$$
2. $$\binom{n}{1} = n$$
3. $$\binom{n}{0} = 1$$
4. $$\binom{n}{n-1} = n$$
5. $$\binom{n}{r} = \binom{n}{n-r}$$

Pascal's Formula

Pascal's formula:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Using Pascal's triangle, we can compute $\binom{n}{r}$ recursively.

Example: : Compute $\binom{3}{2}$

$$\begin{aligned}\binom{3}{2} &= \binom{2}{1} + \binom{2}{2} \\ &= \binom{1}{0} + \binom{1}{1} + \binom{2}{2} \\ &= 1 + 1 + 1 \\ &= 3\end{aligned}$$