Lecture 7

188 200
Discrete Mathematics and Linear Algebra

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Overview

Topics for today:

- Introduction to combinatorics
- Basic of counting
- Product rule
- Sum rule
- Difference rule
- Permutation
- Inclusive-exclusive rule
- Tree diagram

Reference : Section 6.1 - 6.3

What is combinatorics?

Combinatorics is the study of arrangements of discrete objects.

Many applications throughout computer field:

- Algorithm complexity analysis
- Resource allocation & scheduling
- Security analysis
- and many more

Today, we will learn the **basics of counting**. Specifically, we will see several simple rules that can be used to solve many combinatoric problems.

Basic of Counting

- You might ask "how hard can counting be?"
  - Isn't it as easy as count "$1, 2, 3, \ldots$"?
- The answer is yes and no.
- In this lecture, we will learn **not how to count but rather what to count**.
- Formally, **counting** is the mathematical action of repeatedly adding (or subtracting or in other words, counting down) one, usually to find out how many objects there are.

Motivating Example

Example: Tossing two coins and observing how many times both coins are head.

There are four possible outcomes:

- HH, HT, TH, TT
- Each outcome is equally likely to occur.

A sample space (S) is the set of all possible outcomes.

- $S = \{$HH, HT, TH, TT$\}$

An event (E) is a subset of a sample space

- $E = \{$HH$\}$

Probability of E is

$$P(E) = \frac{|E|}{|S|}$$

The Product Rule

The product rule applies when a problem can be divided into multiple tasks.

Suppose a process can be broken into a sequence of k tasks. And there are n_1 ways to do task 1, n_2 ways to do task 2, ..., and n_k ways to do task k :

- Then there are $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$ ways to complete the problem.
- Also called **the multiplication rule**

Let's see some example

Product Rule Example

Example: Suppose there are 18 math major students and 325 CS major students. How many ways are there to pick one math major student and one CS major student?

Step 1: Choose one math major student.

- There are 18 possible ways.

Step 2: Choose one CS major student.

- There are 325 possible ways.

By the product rule: There are $18 \cdot 325 = 5850$ ways to choose one Math major student and one CS major student.

Product Rule Examples 2

Example: How many strings of length 5 that start with even number and end with odd number?

Step 1: Choose the first digit.
- There are 5 possible ways.

Step 2: Choose the second digit.
- There are 10 possible ways.

Step 3: Choose the third digit.
- There are 10 possible ways.

Step 4: Choose the forth digit.
- There are 10 possible ways.

Step 5: Choose the fifth digit
- There are 5 possible ways.

By the product rule: There are $5 \cdot 10 \cdot 10 \cdot 10 \cdot 5 = 25000$ valid strings.

Product Rule Example 3

Example: Suppose in one province in Thailand, license plates consist of two letters followed by four digits. The first digit can only be number 2, 4 or 6. How many valid license plates are there?

Step 1: Choose the first letter: There are 44 ways.

Step 2: Choose the second letter: There are 44 ways.

Step 3: Choose the first digit: There are 3 ways.

Step 4: Choose the second digit: There are 10 ways.

Step 5: Choose the third digit: There are 10 ways.

Step 6: Choose the fourth digit: There are 10 ways.

By the product rule: There are $44 \cdot 44 \cdot 3 \cdot 10 \cdot 10 \cdot 10$ $= 5,808,000$ valid licenses.

The Sum Rule

The sum rule applies when a single task can be completed using several different approaches.

Suppose that a single task can be completed in either one of n_1 ways, one of n_2 ways, ..., or one of n_k ways. Then the task can be completed in $n_1 + n_2 + ... + n_k$ different ways.

Also called **the additional rule**.

To apply the sum rule, we break the set of all possible solutions to the problem into disjoint subsets. E.g., if we have k types of solutions, then $S = S_1 \cup S_2 \cup \ldots \cup S_k$:

$$\begin{aligned} |S| &= |S_1 \cup S_2 \cup \ldots \cup S_k| \\ &= |S_1| + |S2| + \ldots + |Sk| \quad \text{Since } S_1, \ldots, S_K \text{ are disjoint.} \\ &= n_1 + n_2 + \ldots + n_k \end{aligned}$$

The Sum Rule: Example

Example: Kaew wants to travel from Khon Kaen to Bangkok.

- If she flies there are 2 flights to Bkk.
- If she takes a bus, there are 24 different buses to choose.
- If she takes a train, there are 6 different trains to choose.

1. How many ways can she travel to BKK?
2. What is a probability that she will take a train?

By the sum rule: There are $2 + 24 + 6 = 32$ ways.

The probability that she will a train is

$$\frac{6}{32} = 0.1875.$$

The Sum Rule: Example 2

Example: How many three-digit integer (integer from 100 to 999) are divisible by 5?

Solution

The Difference Rule

One consequence from the sum rule is **different rule**.

Suppose a counting problem E can be broken down to two subsets B and $E - B$ such that B and $N - B$ are disjoint. Then

$$N_{E-B} = N_E - N_B$$

Example: Suppose that a password for a certain system is made from exactly four symbols. Each symbol can be chosen from 26 uppercase letters or 10 digits.

1. How many passwords are there if repetition is allowed?
2. How many passwords are there if repetition is not allow?
3. How many passwords contain repeated symbols?
4. What is a probability that a password contains a repeated symbol?

The Difference Rule: Solution

Solution:

1. There are 36 possible ways to choose for each symbol.
   - $36 \cdot 36 \cdot 36 \cdot 36 = 1,679,616.$
2. If repetition is not allowed, there are 36, 35, 34, 33 ways to choose the first, second, third and fourth symbol respectively.
   - $36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720$.
3. **By the difference rule:** number of passwords with repeated symbol is equal to number of all password minus number of password without repeated symbol.
   - 1,679,616 - 1,413,720 = 265,896
4. The probability is $265,896/1,679,616 \approx 0.158$

Probability of the complement of an event

If S is a finite space and A is an event in S, then **probability of the complement** of A

$$P(\overline{A}) = 1 - P(A)$$

This can be illustrated by the previous example:

What is the probability of a password contains a repeated symbol?

- Let A be “choosing password contains repeated symbol”
- Then $\overline{A}$ will be ”choosing password contains no repeated symbol”
- $P(\overline{A}) = 1,413,720/1,679,616 \approx 0.842$
- By the probability of complement $P(A)$ is $1 - 0.842 \approx 0.158$.

Permutation

A permutation is an ordering of the object in a row.

In general, we can use the product rule to count the number of permutations of a given set.

Given a set of n items, we have:

- n ways to pick the 1st item in the permuted set
- $n-1$ ways to pick the 2nd item in the permuted set
- $n-2$ ways to pick the 3rd item in the permuted set
- $\ldots$
- 1 way to choose the last item in the permuted set

So, for a set of size n, we have $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$ ways to permute that set

Permutation: Example

Example: Suppose you are given a word “COMPUTER”

- What is the number of permutation of the word?
- What is the number of permutation of the word if the letter “CO” is fixed as a unit?
- What is a probability that a word permuted from “COMPUTER” contain “CO” as a unit?

Solution:

Permutation: Example 2

Example: At a meeting, there are six people sitting around a circular table. How many different way can people be seated?

Note: It doesn't matter who sit in which chair, but it matter who sit next to whom.

Solution:

r-Permutation

Often we are interested in arranging subset of a given set.

An ***r*-permutation** is a permutation of a subset with r elements.

An r-permutation of a set n elements is

$$P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$$

or equivalently

$$P(n, r) = \frac{n!}{(n-r)!}$$

r-Permutation: Example

Example:

1. How may ways can 3 of the letters of the word "BYTES" be permuted?
2. How many permutations if the first letter must be "B"?

Solution:

r-Permutation: Example 2

An IP address is a 32-bit string that is used to identify a computer that is connected to the Internet. There are three categories of IP addresses that can be assigned to computers:

1. Class A addresses start with "0" followed by a 7- bit network ID and a 24-bit host ID
2. Class B addresses start with "10" followed by a 14-bit network ID and a 16-bit host ID
3. Class C addresses start with "110" followed by a 21-bit network ID and an 8-bit host ID

Note:

- 1111111 cannot be used at the network ID of class A.
- Host IDs consisting of only 1s or only 0s cannot be used.

How many valid P addresses are there?

Example 2 cont.

Solution:

More Complex Counting Problems

Example: Consider a wedding picture of 6 people

– There are 10 people, including the bride and groom

How many possibilities are there if the bride must be in the picture?

Solution:

The Wedding Photo Example 2

Example: How many possibilities are there if the bride and groom must both be in the picture?

Solution:

The Wedding Photo Example 3

Example: How many possibilities are there if only one of the bride and groom are in the picture?

Solution:

The inclusion-exclusion principle

When counting the possibilities, we can't include a given outcome more than once!

The inclusion-exclusion rule:

- Recall from set theory $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- And $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

Example: How may bit strings of length eight start with 1 or end with 00?

- 1 X X X X X 0 0

Solution:

Inclusive-exclusive Principle Example cont.

Inclusive-exclusive Principle Example 2

Example: How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?

- e.g. X X X X 0 0 0 0 0 X
- e.g. X 1 1 1 1 1 X X X X

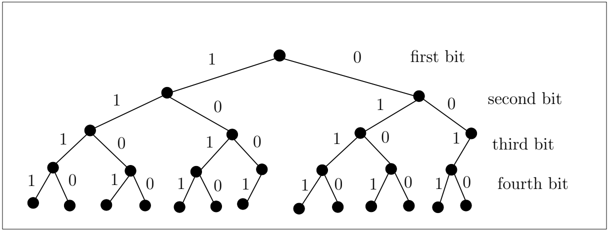
Solution:

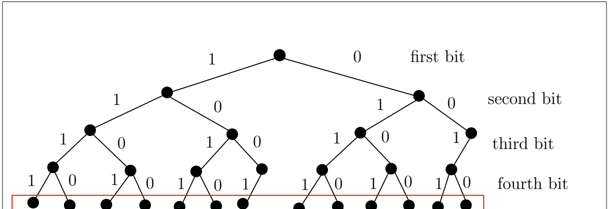
Inclusive-exclusive Principle Example 2 cont.

Tree Diagram

- We can use tree diagrams to enumerate the possible choices
- Once the tree is laid out, the result is the number of (valid) leaves

Example: Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s





Tree Diagram : Example 2

Example: Consider the following tree diagram. How many ways can Liverpool finish with the record 2 wins and 1 loose.