Lecture 6

188 200
Discrete Mathematics and Linear Algebra

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Overview of This Lecture

Recall that a universal statement, i.e., $\forall\ x\ P(x)$ cannot be proved by giving examples.

– For every set with n elements, the cardinality of its power set is 2^n.

- We know that this is true, but how do we prove it?
- If $n = 2$, e.g. $\{1,2\}$, the power set is $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. So the cardinality is $4 = 2^2$.
- But this dose not guarantee that the cardinality will be 2^n when n is another number.

In this lecture, we study one more method of proof that is one of the more recently developed technique, **induction**.

Reference:

- Section 4.2 - 4.4

Basic of Induction

Induction can be informally described as you climbing an infinite stair.

- You have to first step on the first step of the stair.
- Then repeat
  - From your current position, move up one step.

Let's see it in action: prove $P(x)$ for $x \in \{0, 1, 2, 3, 4, \ldots\}$

Proof :

- First, show $P(x)$ is true for $x = 0$.
  - This is corresponding to the first step of the stair.

Basic of Induction 2

- Then, show that if its true for some value $x \geq 0$, then it is true for $x+1$.
  - Show: $P(x) \rightarrow P(x+1)$
  - This is climbing the stairs.
  - Let $x = 0$. Since it's true for $P(0)$ (base case), it's true for $x = 1$.
  - Let x=1. Since it's true for $P(1)$ (previous bullet), it's true for x=2
  - Let x=2. Since it's true for $P(2)$ (previous bullet), it's true for $x = 3$
  - Let $x = 3$.. . . .
  - And onwards to infinity

Thus, we have shown it to be true for all non-negative numbers

Q.E.D.

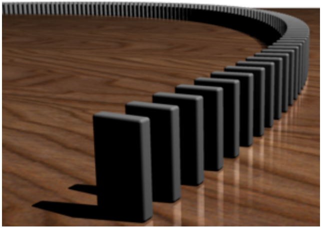
Concept of Induction

An induction is a method of mathematical proof that consists of three parts:

- **Basic Case:** show that a statement is true for an initial value.
- **Induction hypothesis:** assume that a statement is true for an arbitrarily element greater than or equal to the initial value.
- **Inductive step:** show that it is also true for the next element in a sequence.

Induction can also be illustrated by reference to the sequential effect of falling dominoes.

- The first domino falls
- Whenever a domino fall, the next neighbor will also fall.
- So it is concluded that all of the dominoes will fall, and that this fact is inevitable.



Induction Example

Example : Show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for a non-negative integer n. Let $P(n)$ be a statement $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$.

Proof :

- Base case:
- Induction hypothesis:
- Inductive Step:

How does the prove work?

- We show the base case : $P(0)$.
- Then we show that if $P(k)$ is true, then $P(k+1)$ is true for $k \geq 0$, i.e., $P(k) \rightarrow P(k+1)$.

- We know its true for $P(0)$
- Because of $P(k) \rightarrow P(k+1)$, if its true for $P(0)$, then its true for $P(1)$.
- Because of $P(k) \rightarrow P(k+1)$, if its true for $P(1)$, then its true for $P(2)$.
- Because of $P(k) \rightarrow P(k+1)$, if its true for $P(2)$, then its true for $P(3)$.
- And onwards to infinity.

Thus, it is true for all possible values of n.

In other words, we showed that:

$$[P(0) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n\ P(n)$$

Induction Example 2

Example 2: The sum of the first n odd integers is n^2.

- e.g., for $n = 5$, $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
- Mathematically, let $P(n)$ be a statement $\sum_{j=1}^{n} 2j - 1 = n^2$.

Proof :

Summarizing Induction Method

- Identify a property that need to be shown true, i.e. $P(n)$
  - e.g. Let $P(n)$ be a statement $\sum_{j=1}^{n} 2j - 1 = n^2$
- Show that $P(a)$ is true.
  - a is an initial value, usually $a = 0$.
  - Basis step
- Suppose that $P(k)$ is true for any integer $k \geq a$.
  - Induction hypothesis.
- Show that $P(k+1)$ is true.
  - Inductive step

Example 3

Example 3 : The sum of the first n positive even integers is $n^2 + n$.

- e.g., for $n = 5$, $2 + 4 + 6 + 8 + 10 = 30 = 5^2 + 5$
- Mathematically, let $P(n)$ be a statement $\sum_{j=1}^{n} 2j = n^2 + n$.

Proof :

Hint on Induction Proof

We try to manipulate the $(k+1)$ case to try to make part of it to look like the k case, i.e., the induction hypothesis.

$$P(k+1) = \underbrace{2+4+6+\ldots+2k}_{P(k)} + 2(k+1)$$

We then replace that part with the induction hypothesis, and try to derive general formula for the $(k+1)$ case.

$$\begin{aligned} P(k+1) &= P(k) + 2(k+1) && \text{By I.H.} \\ &= k^2 + k + 2(k+1) && P(k) = k^2 + k \\ &= k^2 + 3k + 2 && \text{desired result.} \end{aligned}$$

Example 4

Example 4 : Prove that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

- i.e. $1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof:

Example 5

Example 5 : Show that $n! < n^n$ for all integer $n > 1$.

Definition : n **factorial**, denoted $n!$, is defined to be the product of all integers from 1 to n:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

Note that $0! = 1$.

Proof :

Example 6

Example 6 : Prove that for any real number r where $r \neq 1$, and any integer $n \geq 0$:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+2} - 1}{r - 1}.$$

- i.e. $1 + r + r^2 + r^3 + \ldots + r^n = \frac{r^{n+2}-1}{r-1}$.
- This is a summation of a geometric sequence.

Proof :

Example 7

Example 7 : Show that for all integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

- "b is divisible by a" is denoted $a \mid b$.
- $3 \mid (2^{2n} - 1)$ for all integer $n \geq 1$.
- $a \mid b$ implies $\exists$ integer k such that $b = k \cdot a$.

Proof :

Example 8

Example 8 : Suppose we are given a sequence $a_1, a_2, a_3, \ldots$ which is defined as

- $a_1 = 2$
- $a_k = 5a_{k_1}$ for all integers $k \geq 2$

Prove that the sequence satisfy the property $a_n = 2 \cdot 5^{n-1}$ for all integers $n \geq 1$.

Proof :

Example 9

Example 9 : Let's suppose there are coins for 3 and 5 "strang". Prove that n strang can be obtained using 3-strang and 5-strang coins, for any integers $n \geq 8$.

- 8 strang = 1 x 3-strang + 1 x 5-strang
- 17 strang = 4 x 3-strang + 1 x 5-strang

Proof:

Strong Induction

- Non-strong mathematical induction assumes $P(k)$ is true, and uses that (and only that!) to show $P(k+1)$ is true
- **Strong mathematical induction assumes $P(1)$, $P(2)$, $\ldots$, $P(k)$ are all true, and uses that to show that $P(k+1)$ is true.**

$$[P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$$

- Sometimes **strong induction** is easier to use.
- It can be shown that strong induction and induction are equivalent:
  - any proof by induction is also a proof by strong induction
  - any proof by strong induction can be converted into a proof by induction
- Strong induction also referred to as **complete induction**; in this context induction is referred to as **incomplete induction**.

Strong Induction Example

Example : Show that any integer number greater than 1 can be written as the product of one or more prime numbers.

Proof :

- Let $P(n)$ be a statement "n can be written as the product of one or more prime number".
- **Base case:** for $n = 2$

  – 2 is the product of 2 (remember that 1 is not a prime number).
- **Induction hypothesis:** assume that $P(2)$, $P(3)$,...,$P(k)$ are all true.

  • $2, 3, 4, \ldots, k$ can be written as the product of one or more prime number.

  • $3 = 3,\ 4 = 2 \cdot 2,\ 5 = 5,\ 6 = 3 \cdot 2 \ldots$
- **Inductive step:** show that $P(k+1)$ must also be true.

Strong Induction Example cont.

Inductive Step : there are two cases to be considered:

- $k + 1$ is prime
  - Then $(k + 1)$ can be written as the product of itself.

- $k + 1$ is not prime (composite)
  - Then $k + 1$ can be written as the product of a and b, where $2 \leq a, b < k + 1$.
  - By the induction hypothesis, both $P(a)$ and $P(b)$ must be true.
    - I.H. : $P(2)$, $P(3)$, $P(4)$, $\ldots$, $P(k)$ are all true.
  - Therefore $k + 1$ can be written as the product of prime numbers..

Q.E.D.

Strong Induction Example 2

Example : Consider the game where there are 2 piles of n matches on each pile. Two players take turns removing any number of matches they want from one of the two piles. The player who removes the last match wins the game. Show that the second player can always guarantee a win.

What strategy could the second player use?

Let $P(n)$ be a statement "the second player always win for any positive number of n".

Proof :

Strong Induction Example 3

Example : Consider the following process:

- Take a pile of n stones
  - Split the pile into two smaller piles of size r and s.
  - Repeat until you have n piles of 1 stone each.
- Take the product of all the splits
  - So all the r's and s's from each split.
- Sum up each of these products.
- Prove that this product equals $\frac{n(n-1)}{2}$

Proof:

Strong induction VS. Non-strong Induction

We will solve the same problem with both strong and non-strong induction to some comparison.

Example : For any postage total amount greater than or equal to 20, we can determine which amount of postage can be written with 5 and 6 cent stamps.

- For $n = 25$, $P(25) = 5 + 5 + 5 + 5 + 5$.
- For $n = 33$, $P(33) = 6 + 6 + 6 + 5 + 5 + 5$.

Proof : We will prove this by non-strong induction first.

Prove by Strong Induction

Now let's prove the previous example with strong induction.

Proof :

Notes on Strong Induction VS. Induction

- From the previous example, which one is easier?
  - It really depend on the problem.
- You can think of strong induction as the generalization of non-strong induction.
  - You can derive strong induction from non-strong induction. We will omit this.