Lecture 5

188 200
Discrete Mathematics and Linear Algebra

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen Uiversity

June 25, 2009

Overview

Today we will learn about Sequence and Recursion.

- Sequence
- Summation, Product notation, Factorial notation
- Recursion
- Arithmetic sequence, Geometric sequence
- Tower of Hanoi, Fibonacci sequence
- Determining an explicit formula by a second-order linear homogeneous recurrence relation technique

Reference

- Section 4.1, Section 8.1 - 8.3

Sequence

Sequence is an ordered list of elements.

- sequence is denoted as

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n(, \ldots)$$

  - can be finite or infinite.
- each a_k (read "a sub k" or simply "a k") is called "**term**".
- k is called "**index**".
- a_m is called an "**initial term**".
- m usually starts from 0 or 1.
- For example $0, 3, 6, 9, \ldots$
- an **explicit formula** (or **general formula**) of a sequence is a rule that shows how the values of a_k depends on k.

  — for example $a_k = 3k$ for all integers $k \geq 0$

Finding Terms of Sequences from Explicit Formulas

Example: Compute the first three terms of the following explicit formula:

- $a_k = \frac{k}{k+1}$ for all integers $k \geq 1$
- $b_i = \frac{i-1}{i}$ for all integers $i \geq 2$

Solution

$$a_1 = \frac{1}{1+1} = \frac{1}{2}$$
$$a_2 = \frac{2}{2+1} = \frac{2}{3}$$
$$a_3 = \frac{3}{3+1} = \frac{3}{4}$$

$$b_2 = \frac{2-1}{2} = \frac{1}{2}$$
$$b_3 = \frac{3-1}{3} = \frac{2}{3}$$
$$b_4 = \frac{4-1}{4} = \frac{3}{4}$$

Finding an Explicit Formula for Sequences

Example: Find an explicit formula for the following sequence:

$$1, \ -\frac{1}{4}, \quad \frac{1}{9}, \ -\frac{1}{16}, \quad \frac{1}{25}, \ -\frac{1}{36}, \ldots$$

Solution

- The sequence can be written as $\frac{1}{1^2}, -\frac{1}{2^2}, \frac{1}{3^2}, -\frac{1}{4^2}, \frac{1}{5^2}, -\frac{1}{6^2}, \ldots$
  - $a_k = \frac{\pm 1}{k^2}$ for every integers $k \geq 1$.
- The sign of the sequence alternates between positive and negative.
  - positive with odd term and negative with even term.
  - $(-1)^{k+1}$
- Thus the explicit formula is $a_k = \frac{(-1)^{k+1}}{k^2}$ for $k \geq 1$.
  - To start k with 0, $a_k = \frac{(-1)^k}{(k+1)^2}$ for $k \geq 0$.

Summation

The summation is denoted by

$$\sum_{k=m}^{n} a_k$$

- "the summation from $k = m$ to n of a_k".
- **The expanded form**: $a_m + a_{m+1} + a_{m+2} \ldots + a_n$
- m is lower limit and n is upper limit.
- $\sum_{k=1}^{5} k + 1 = 2 + 3 + 4 + 5 + 6 = 20$
- $\sum_{k=1}^{5} 3 = 3 + 3 + 3 + 3 + 3 = 15$
- $\sum_{k=0}^{4} (2^{k+1} - 2^k) =$
  $(2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + (2^4 - 2^3) = 15$

Finding Sum Notation of Expanded Forms

Example: Find the summation notation of the following expanded form

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \frac{3}{n+2} + \ldots + \frac{n+1}{2n}$$

Solution: The sequence can be listed as:

$$\frac{0+1}{n+0} + \frac{1+1}{n+1} + \frac{2+1}{n+2} + \frac{3+1}{n+3} + \ldots + \frac{n+1}{n+n}$$

The general term of this summation then can be expressed as $\frac{k+1}{n+k}$ for integers k from 0 to n.

Hence the summation notation is

$$\sum_{k=0}^{n} \frac{k+1}{n+k}.$$

Product Notation

The product of a sequence is denoted as

$$\prod_{k=m}^{n} a_k$$

- the product from $k = m$ to n of a_k.
- $a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_n$

For example:

- $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$
- $\prod_{k=1}^{4} \frac{k}{k+1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$

Factorial Notation

For $n \geq 1$, n **factorial** (denoted as $n!$) is the product of all the integers from 1 to n

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

Note that **zero factorial** ($0!$) is defined to be 1.

For example:

- $2! = 2 \cdot 1 = 2$
- $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
- $\frac{n!}{(n-3)!} = n \cdot (n-1) \cdot (n-2)$

Properties of Summations and Products

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

2. For some constant c
$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$

3. $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$$

Example Using Property of Summations and Products

Example: Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k, write each of the following as a single summation or product.

- $\sum_{k=m}^{n} a_k + 2\sum_{k=m}^{n} b_k$

$$
\begin{aligned}
\sum_{k=m}^{n} a_k + 2\sum_{k=m}^{n} b_k &= \sum_{k=m}^{n}(k+1) + 2\sum_{k=m}^{n}(k-1) && \text{by substitution} \\
&= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2(k-1) && \text{by property 2} \\
&= \sum_{k=m}^{n}((k+1) + 2(k-1)) && \text{by property 1} \\
&= \sum_{k=m}^{n}(3k-1) && \text{by simplification}
\end{aligned}
$$

Changing Variable of Summations

Example: Transforming the following summation

$$\sum_{k=0}^{6} \frac{1}{k+1}$$

by changing $j = k + 1$ (i.e. change index to 1 to 7).

Solution:

- First calculate the new lower and upper limit
  - when $k = 0$, $j = 1$
  - when $k = 6$, $j = 7$
- Since $j = k + 1$, $\frac{1}{k+1} = \frac{1}{j}$
- Thus

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}$$

Changing Variable : More Example

Example: Write the following summations as a single summation.

$$3 \cdot \sum_{k=-1}^{n-1} (2k+3) + \sum_{k=2}^{n+2} (4-5k)$$

Solution: First change both summations to the same index.

- First summation: Let $j = 0$, then $j = k + 1$.
  - when $k = n - 1$, $j = n$.
  - $2k + 3 = 2(j - 1) + 3 = 2j + 1$.
- Second summation: Let $j = 0$, then $j = k - 2$.
  - when $k = n + 2$, $j = n$.
  - $4 - 5k = 4 - 5(j + 2) = -6 - j$.

$$3 \cdot \sum_{j=0}^{n} (2j+1) + \sum_{j=0}^{n} (-6-j) = \sum_{j=0}^{n} (5j-3)$$

Recursion

Recursion can be used to defined sequence.

- **A recurrence relation:** a formula that relates each term a_k to some previous terms $a_{k-1}, a_{k-2}, \ldots$
  - $a_k = a_{k-1} + 2a_{k-2}$
- **The initial conditions:** the values of $a_0, a_1, \ldots$

Example: For all integers $k \geq 2$, find the terms b_2, b_3 and b_4:

- $b_k = b_{k-1} + k \cdot b_{k-2} + 1$ (recurrence relation)
- $b_0 = 1$ and $b_1 = 2$ (initial conditions)

Solution:

$$b_2 = b_1 + 2b_0 + 1 = 2 + 2 \cdot 1 + 1 = 5$$
$$b_3 = b_2 + 2b_1 + 1 = 5 + 3 \cdot 2 + 1 = 12$$
$$b_4 = b_3 + 2b_2 + 1 = 12 + 4 \cdot 5 + 1 = 33$$

Arithmetic Sequence

A sequence is called arithmetic sequence iff each term equals the previous term plus a fixed constant.

Formally, a sequence is called arithmetic sequence iff there is a constant d such that its recurrence relation is

$$a_k = a_{k-1} + d \qquad \text{for all integers } k \geq 1$$

Or equivalently,

$$a_n = a_0 + d \cdot n \qquad \text{for all integers } n \geq 0$$

Arithmetic Sequence: Example

Example:

- Assume there is no air resistant, a ball will fall 9.8 meter each second.
- The ball falls 4.8 meter between 0 and 1 second.
- How far will the ball fall between 60 and 61 second?

Solution:

- Let d_k be the distance that a ball would fall between k and $k+1$ second.
- Thus $d_k = d_{k-1} + 9.8$ for $k \geq 1$.
- Equivalently, $d_n = d_0 + n \cdot 9.8$ for $n \geq 0$.
- And $a_0 = 4.8$
- Hence, $d_{60} = 4.8 + 80 \cdot 9.8 = 592.8$ meters.

Geometric Sequence

A sequence is called a geometric sequence iff each term equals the previous term times a fixed constant.

Formally, sequence is called geometric sequence iff there is r such that its recurrence relation are

$$a_k = r \cdot a_{k-1} \qquad \text{for} \quad k \geq 1.$$

Or equivalently

$$a_n = a_0 \cdot r^n \qquad \text{for} \quad n \geq 0.$$

Geometric Sequence: Example

Example: Consider the following situation:

- A bank pays interest rate of 4% per year.
- Suppose the initial deposit is 100,000 baht.
- How much money will there be at the end of year 21?

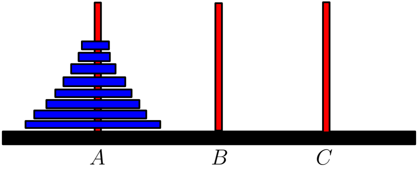
Solution:

- Let a_k be the amount of money at the end of year k.
- Money end of k = Money end of $k-1$ + interest gained in k.
  - $a_k = a_{k-1} + 0.04 \cdot a_{k-1} = (1.04)a_{k-1}$.
- Equivalently, $a_n = a_0 \cdot (1.04)^n$.
- Hence $a_{21} = (100,000)(1.04)^{21} \approx$ 227,876.81 baht.

The Tower of Hanoi

The tower of Hanoi is a puzzle that consists of:

- k disks of woods of different sizes
- 3 poles: A, B and C



Rules: • move a disk one at a time
• never place a larger disk on top of a smaller one

You want to move all disks from A to C.

How many moves will it take if there are 8 disks?

Can you show how to move 1 disk? 2 disks? 3 disks?

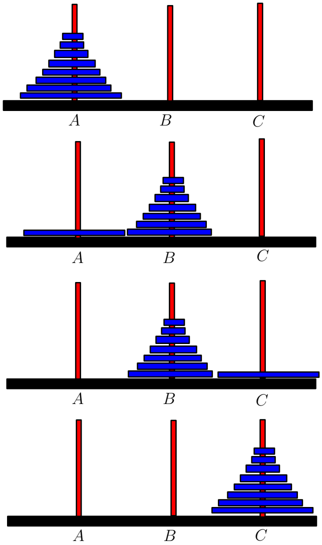
The Tower of Hanoi 2

Let's use recursion!

Consider moving k disks from A to C:

1. Moving the top $k-1$ disks from A to B.
2. Moving the bottom disk from A to C.
3. Moving the top $k-1$ disks from B to C.

We will assume that we know how to move $k-1$ disks from A to B.



The Tower of Hanoi 4

Hence,

# of moves of k from A to B

$=$ **# of moves of $k-1$ from A to B**

$+$ **# of moves of the bottom disk from A to C**

$+$ **# of moves of $k-1$ from B to C**

Let m_k be the minimum number of moves needed to transfer a tower of k disks from one pole to another. Hence

$$m_k = m_{k-1} + 1 + m_{k-1} = 2m_{k-1} + 1$$

And $m_1 = 1$.

The Tower of Hanoi 5

So how many moves does it take to move 8 disks?

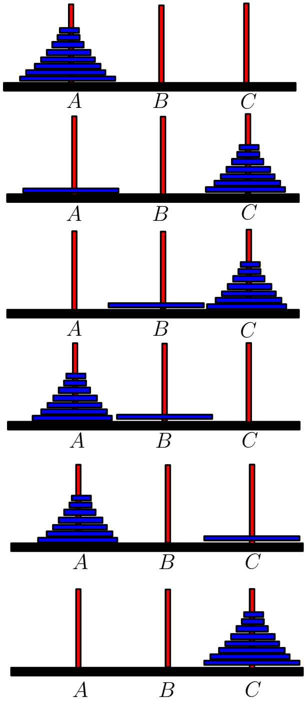
$$
\begin{aligned}
m_1 &= 1 \\
m_2 &= 2 \cdot m_1 + 1 = 2 \cdot 1 + 1 = 3 \\
m_3 &= 2 \cdot m_2 + 1 = 2 \cdot 3 + 1 = 7 \\
m_4 &= 2 \cdot m_3 + 1 = 2 \cdot 7 + 1 = 15 \\
m_5 &= 2 \cdot m_4 + 1 = 2 \cdot 15 + 1 = 31 \\
m_6 &= 2 \cdot m_5 + 1 = 2 \cdot 31 + 1 = 63 \\
m_7 &= 2 \cdot m_6 + 1 = 2 \cdot 63 + 1 = 127 \\
m_8 &= 2 \cdot m_7 + 1 = 2 \cdot 127 + 1 = 255
\end{aligned}
$$

The Tower of Hanoi 3

Is this the most efficient?

Yes! Because:

1. Moving the bottom disk from A to C, first move the top $k-1$ disks out.
2. Instead move $k-1$ to C, at least two extra steps.
   - Move $k-1$ from A to C.
   - Move $k-1$ from C to A.



Fibonacci Sequence

Consider the sequence where

- **Each term is the sum of the previous two terms**.
- **Recurrence relation** is $a_k = a_{k-1} + a_{k-2}$, where the first two terms are 1, i.e. $a_1 = a_2 = 1$.
- $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
- This is the **Fibonacci sequence**

Reproducing Rabbits

You have one pair of rabbits (male and female) on an island:

- Suppose the following statements are true:
  - Every rabbit pairs pregnant for one month.
  - They start pregnant after they are one month old.
  - They give birth to another pair every month.
- This process repeats indefinitely
  - Assume that every pairs live forever (no rabbit will die).

How many rabbit pairs are there after 10 months?

Reproducing Rabbits 2

- **Starting: 1 pair**
  - The original pair was born.
- **End of first month: 1 pair**
  - The original (and now pregnant) pair
- **End of second month: 2 pairs**
  - One new pair and one original pair
- **End of third month: 3 pairs**
  - One new pair, and two old pairs
- **End of forth month: 5 pairs**
  - 2 new pairs plus 3 old pairs
- **End of fifth month: 8 pairs**
  - 3 new pairs and 5 old pairs

Reproducing Rabbits 3

- **End of sixth month: 13 pairs**
  - 5 new rabbit pairs and 8 old pairs
- **End of seventh month: 21 pairs**
  - 8 new rabbit pairs and 13 old pairs
- **End of eighth month: 34 pairs**
  - 13 new rabbit pairs and 21 old pairs
- **End of ninth month: 55 pairs**
  - 21 new rabbit pairs and 34 old pairs
- **End of tenth month: 89 pairs**
  - 34 new rabbit pairs, and 55 old pairs

Note the sequence: $\{ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \}$

The Fibonacci sequence again!

Solving Reproducing Rabbits with Recursion

Observation: number of rabbit pairs born at the end of month k is the same as the number of pairs alive at the end of month $k-2$.

Because rabbits pairs born at the end of $k-2$ give birth at the and of month k.

Thus,

$\#$ **rabbit pairs alive at end month** k

$=$ $\#$ **rabbit pairs alive at end of month** $k-1$

$+$ $\#$ **rabbit pairs born at end of month** k

$=$ $\#$ **rabbit pairs alive at end of month** $k-1$

$+$ $\#$ **rabbit pairs alive at end of month** $k-2$

Solving Reproducing Rabbits with Recursion 2

Let a_k be the number of rabbit pairs alive at the end of month k.

$$a_k = a_{k-1} + a_{k-1}$$

We know $a_0 = a_1 = 0$.

So what is a_{10}? (How many pairs alive at the end of month 10?)

$$\begin{aligned}
a_2 &= a_1 + a_0 = 1 + 1 = 2 \\
a_3 &= a_2 + a_1 = 2 + 1 = 3 \\
a_4 &= a_3 + a_2 = 3 + 2 = 5 \\
a_5 &= a_4 + a_3 = 5 + 3 = 8 \\
a_6 &= a_5 + a_4 = 8 + 5 = 13 \\
a_7 &= a_6 + a_5 = 13 + 8 = 21 \\
a_8 &= a_7 + a_6 = 21 + 13 = 34 \\
a_9 &= a_8 + a_7 = 34 + 21 = 55 \\
a_{10} &= a_9 + a_{10} = 55 + 34 = 89
\end{aligned}$$

Finding an Explicit Formula

Tower of Hanoi : The recurrence relation is $m_k = 2m_{k-1} + 1$ for $k \geq 2$ and $m_1 = 1$. Find the explicit formula for the tower of Hanoi sequence.

Solution: Recall that

$$m_2 = 2 \cdot 1 + 1 = 2^1 + 1$$
$$m_3 = 2 \cdot (2 + 1) + 1 = 2^2 + 2 + 1$$
$$m_4 = 2 \cdot (2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$$
$$m_5 = 2 \cdot (2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$$

Hence $m_k = 2^{k-1} + 2^{k-2} + \ldots + 2 + 1$.

- And $2m_k = 2^k + 2^{k-1} + \ldots + 2^2 + 2$.
- So $2m_k - m_k = 2^k - 1$.

Therefore the explicit formula is $a_k = 2^k - 1$.

Finding Sequence that Satisfies Recurrence Relation

A second order linear homogeneous recurrence relation with constant coefficient is a recurrence relation of the form

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$$

for $k \geq$ some fixed integer, A and B are constant and $B \neq 0$.

The sequence $1, t, t^2, t^3, \ldots$ satisfies a_k where t is a nonzero real number iff t satisfies the following equation:

$$t^2 - At - B = 0$$

$t^2 - At - B = 0$ is called the **characteristic equation** of the relation.

Finding Sequence that Satisfies Relation : Example

Example: Consider the following recurrence relation

$$a_k = a_{k-1} + 2a_{k-2} \text{ for all } k \geq 2$$

Find sequences that satisfie the relation.

Solution:

- The relation is second linear homogeneous where $A = 1$ and $B = 2$.
- The sequence $1, t, t^2, t^3, \ldots$ satisfies a_k iff $t^2 - At - B = 0$
  - $t^2 - t - 2 = 0$.
- Solve for t, $t = 2$ and -1.
- Hence the following sequences satisfy the relation

$$1, 2, 2^2, 2^3, 2^4, \ldots$$

$$1, -1, (-1)^2, (-1)^3, \ldots$$

Finding an Explicit Formula Using Characteristic Equation

For a second-order linear homogeneous recurrence relation

$$a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$$

If its characteristic equation has two distinct solutions, r and s, then the explicit formula of the recurrence is

$$a_k = C \cdot r^k + D \cdot s^k$$

where C and D can be computed from a_0 and a_1.

Finding an Explicit Formula : Example

Example: Find an explicit formula of $a_k = a_{k-1} + 2a_{k-2}$ for all $k \geq 2$, with initial conditions $a_0 = 1$ and $a_1 = 8$.

Solution: From the previous example the solutions to the characteristic equation are 2 and -1. Because they are distinct ($2 \neq -1$), an explicit formula is $a_k = C \cdot r^k + D \cdot s^k$.

- When $a_0 = 1$, $1 = C \cdot 2^0 + D \cdot (-1)^0$
  - $1 = C + D$
- When $a_1 = 8$, $8 = C \cdot 2^1 + D \cdot (-1)^1$
  - $8 = 2C - D$

Solving both equations, $C = 3$ and $D = -2$.

Therefore $a_k = 3 \cdot 2^k + (-2) \cdot (-1)^k$.

- $a_k = 3 \cdot 2^k - 2 \cdot (-1)^k$

Summarizing How to Find an Explicit

1. Determine A and B from a second order relation.
   - $a_k = A \cdot a_{k-1} + B \cdot a_{k-2}$
2. Solve it's characteristic equation, say r and s.
   - $t^2 - A \cdot t - B = 0$
3. If $r \neq s$, we solve an explicit formula $a_k = C \cdot r^k + D \cdot s^k$ for C and D.
   - By substitute a_0 and a_1.
4. Substitute C and D back to obtain an explicit formula a_k.
   - $a_k = C \cdot r^k + D \cdot s^k$

Explicit Formula for Fibonacci Sequence

Consider the Fibonacci sequence

$$a_k = a_{k-1} + a_{k-2} \text{ for } k \geq 2$$

with $a_0 = a_1 = 1$. Find an explicit formula.

Solution:

- The Fibonacci sequence is second-order linear homogeneous recurrence relation with $A = B = 1$.
- The characteristic equation is $t^2 - t - 1 = 0$.
- The solution are

$$\frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1-\sqrt{5}}{2}.$$

Explicit Formula for Fibonacci Sequence 2

Hence, an explicit formula is

$$a_k = C\left(\frac{1+\sqrt{5}}{2}\right)^k + D\left(\frac{1-\sqrt{5}}{2}\right)^k$$

for $k \geq 0$.

Next step is to solve for C and D:

- Substitute a_0 : $C + D = 1$
- Substitute a_1 : $C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1$.

Solving both equations, we get

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}.$$

Explicit Formula for Fibonacci Sequence 3

Hence,

$$a_k = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^k + \frac{-(1-\sqrt{5})}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^k$$

Or equivalently,

$$a_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \quad \text{for } k \geq 2.$$

- $\{$ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... $\}$?