Lecture 4

188 200
Discrete Mathematics and Linear Algebra

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Overview

In this lecture, we will study one of the most fundamental concepts in mathematics, **set**. We will learn :

- Definition of set
- Properties of set
- The empty set, set partitions and power sets
- Proving set identities
- Russell's paradox and Halting problem.

References

- Chapter 5 : 5.1 - 5.4

So what is a set?

A set is a collections of distinct "objects":

- Some students in class: { Nui, Man, Nat }
- Classes offered by the department: { 188 200, 188 220, . . . }
- Colors of a rainbow: {red, orange, yellow, green, blue, purple }
- Fruits in a market: {mango, lychee, Durian,...}
- Sets can contain non-related elements: { Nui, 188 200, red, mango }

Although a set can contain (almost) anything, we will most often use sets of numbers :

- All positive numbers less than or equal to 5: {1, 2, 3, 4, 5}
- A few selected real numbers: { 2.1, π, 0, -6.32, e }

Set Properties 1

Sets are notated with curly brackets, i.e., $\{\ldots\}$.

Order does not matter in set

- $\{1, 2, 3, 4, 5\}$ is equivalent to $\{3, 5, 2, 4, 1\}$
- However, we often write them in order because it is easier for humans to understand.

Set Properties 2

Sets do not have duplicate elements

- Consider the set of vowels in the alphabet.

  – {a, a, a, e, i, o, o, o, o, o, u} is the same as {a, e, i, o, u}

- Consider the list of students in this class

  – Again, it does not make sense to list somebody twice (unless two people have the same name)

Specifying a Set 1

Sets are usually represented by a capital letter. (A, B, S, etc.)

Elements are usually represented by an italic lower-case letter (a, x, y, etc.)

Easiest way to specify a set is to list all the elements:
$A = \{1, 2, 3, 4, 5\}$

– Not always feasible for large or infinite sets

Specifying a Set 2

Can use an ellipsis (...) $B = \{0, 1, 2, 3, \ldots\}$

- Can cause confusion. Consider the set $C = \{3, 5, 7, \ldots\}$. What comes next?
- If the set is all odd integers greater than 2, it is 9
- If the set is all prime numbers greater than 2, it is 11

Can use set-builder notation

- $D = \{x \mid x \text{ is prime and } x > 2\}$
- $E = \{x \mid x \text{ is odd and } x > 2\}$
- The vertical bar means "such that".
- Thus, set D is read (in English) as: "all elements x such that x is prime and x is greater than 2".

Specify a Set 3

A set is said to contain the various members or elements that make up the set.

– If an element a is a member of (or an element of) a set S, we then use notation $a \in S$

- $4 \in \{1, 2, 3, 4\}$

– If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$

- $7 \notin \{1, 2, 3, 4\}$
- Khon Kaen $\notin \{1, 2, 3, 4\}$

Regular Used Sets

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers (a.k.a whole numbers).
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ is the set of rational numbers
  - Any number that can be expressed as a fraction of two integers (where the bottom one is not zero)
- $\mathbb{R}$ is the set of real numbers

The Universal Set

The **Universal Set** (or **the universe of discourse**) is the set of all of elements (or the universe) from which given any set is drawn. The universal set is denoted by U.

- For the set $\{-2, 0.4, 2\}$, U would be the set of real numbers.
- For the set $\{0, 1, 2\}$, U could be the natural numbers (zero and up), the integers, the rational numbers, or the real numbers, depending on the context.
- For the set of the students in this class, U would be all the students in the University (or perhaps all the people in the world).
- For the set of the vowels of the alphabet, U would be all the letters of the alphabet.

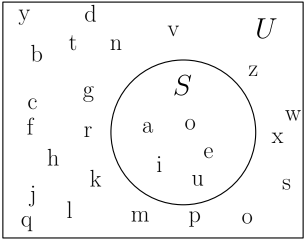
Venn Diagrams

Represents sets graphically

- The box represents the universal set.
- Circles represent the set(s)

Consider set S, which is the set of all vowels in the alphabet.

The individual elements are usually not written in a Venn diagram



Sets of Sets

Sets can contain other sets

- $S = \{\{1\}, \{2\}, \{3\}\}$
- $T = \{\{1\}, \{\{2\}\}, \{\{\{3\}\}\}\}$
- $V = \{\{\{1\}, \{\{2\}\}\}, \{\{\{3\}\}\}, \{\{1\}, \{\{2\}\}, \{\{\{3\}\}\}\}\}$
  - How many elements does V have?

    V has 3 elements!

Note that $1 \neq \{1\} \neq \{\{1\}\} \neq \{\{\{1\}\}\}$

- They are all different.

The Empty Set

If a set has zero elements, it is called **the empty (or null) set**.

- Written using the symbol $\emptyset$
- Thus, $\emptyset = \{\ \}$ $\longleftarrow$ VERY IMPORTANT
- If you get confused about the empty set in a problem, try replacing by $\{\ \}$.

As the empty set is a set, it can be an element of other sets

- $\{\emptyset, 1, 2, 3, x\}$ is a valid set.

Note that $\emptyset \neq \{\emptyset\}$

- The first is a set of zero elements
- The second is a set of 1 element (that one element being the empty set)

Replace $\emptyset$ by $\{\ \}$, and you get: $\{\ \} \neq \{\ \{\ \}\ \}$

- It's easier to see that they are not equal that way

Set Equality

Two sets are equal if they have the same elements

- $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$
  - Remember that order does not matter!
- $\{1, 2, 3, 2, 4, 3, 2, 1\} = \{4, 3, 2, 1\}$
  - Remember that duplicate elements do not matter!

Two sets are not equal if they do not have the same elements

- $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

Subset

If all the elements of a set S are also elements of a set T, then S is a subset of T

- For example, if $S = \{2, 4, 6\}$ and $T = \{1, 2, 3, 4, 5, 6, 7\}$, then S is a subset of T.
- This is specified by $S \subseteq T$.
  - Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$

If S is not a subset of T, it is written as such: $S \nsubseteq T$

- For example, $\{1, 2, 8\} \nsubseteq \{1, 2, 3, 4, 5, 6, 7\}$

Note that any set is a subset of itself!

- Given set $S = \{2, 4, 6\}$, since all the elements of S are elements of S, S is a subset of itself.

Subset 2

The empty set is a subset of all sets (including itself!)

- Recall that all sets are subsets of themselves

All sets are subsets of the universal set.

Another way to define a subset:

- $A \subseteq B : \forall x \mid x \in A \rightarrow x \in B$
- English translation: for all possible values of x, (meaning for all possible elements of a set), if x is an element of A, then x is an element of B.

Proper Subset

If S is a subset of T, and S is not equal to T, then S is a proper subset of T.

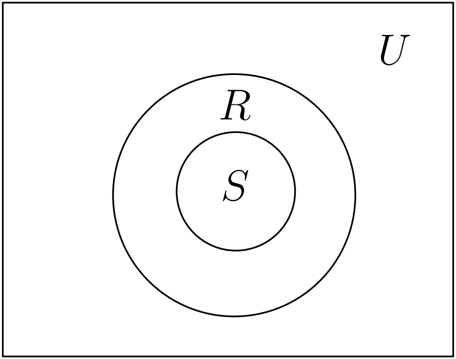
- Let $T = \{0, 1, 2, 3, 4, 5\}$
- If $S = \{1, 2, 3\}$, S is not equal to T, and S is a subset of T
- A proper subset is written as $S \subset T$.
- Let $R = \{0, 1, 2, 3, 4, 5\}$. R is equal to T, and thus is a subset (but not a proper subset) of T
  - Can be written as: $R \subseteq T$ and $R \not\subset T$ (or just $R = T$)
- Let $Q = \{4, 5, 6\}$. Q is neither a subset of T nor a proper subset of T

The difference between subset and proper subset is like the difference between less than or equal to and less than for numbers

The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set).

Proper Subset: Venn Diagram

$$S \subset R$$



Set Carnality

The **cardinality** of a set is the number of elements in a set

- The cardinality of set S is written as $|S|$.

Examples:

- Let $R = \{1, 2, 3, 4, 5\}$. Then $|R| = 5$
- $|\emptyset| = 0$
- Let $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $|S| = 4$

Note: this is the same notation used for vector length in geometry.

A set with one element is sometimes called a singleton set.

Power Sets

Given the set $S = \{0, 1\}$. What are all the possible subsets of S?

- They are: $\emptyset$ (as it is a subset of all sets), $\{0\}$, $\{1\}$, and $\{0, 1\}$.
- **The power set of S (written as $P(S)$) is the set of all the subsets of S.**
- $P(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
  - Note that $|S| = 2$ and $|P(S)| = 4$.

Let $T = \{0, 1, 2\}$.

- Then $P(T) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
  — Note that $|T| = 3$ and $|P(T)| = 8$.
- $P(\emptyset) = \{\emptyset\}$
  — Note that $|\emptyset| = 0$ and $|P(\emptyset)| = 1$

If a set has n elements, then the power set will have 2^n elements.

Tuples

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be elements.

The ordered n-tuple is $(x_1, x_2, \ldots, x_n)$.

- An ordered 2-tuple, (x_1, x_2), is called an ordered pair.
- An ordered 3-tuple, (x_1, x_2, x_3), is called an ordered triple.
- Elements in tuple are not necessarily distinct
  — x_ℓ can be equal to x_m when $\ell \neq m$.
- Two tuples,$(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$, are equal iff, $x_1 = y_1$, $x_2 = x_2$,. . . , $x_n = x_n$.

Order is important in tuples, unlike sets.

Cartesian Products

A **Cartesian product** of A and B is a set of all ordered 2-tuples where each part is from a given set

- Denoted by $A \times B$, and uses parenthesis (not curly brackets).
- For example, 2-D Cartesian coordinates are the set of all ordered pairs $\mathbb{Z} \times \mathbb{Z}$.
  - Recall $\mathbb{Z}$ is the set of all integers.
  - This is all the possible coordinates in 2-D space.
- Example: Given $A = \{a, b\}$ and $B = \{0, 1\}$, what is their Cartiesian product?
  - $C = A \times B = \{(a, 0), (a, 1), (b, 0), (b, 1)\}$

Formal definition of a Cartesian product of A and B:

– $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

A cartesian product of A_1, A_2, ..., A_n:

— $A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A \text{ and} \ldots \; a_n \in A_n\}$

Set Operations : Union

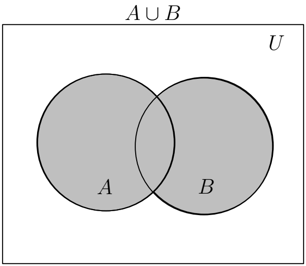
For sets A, B, their union (denoted by $A \cup B$) is the set containing all elements that are either in A, or ($\vee$) in B (or, of course, in both).

Formal definition for the union of two sets:

– $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Further examples

- $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$
- $\{\text{Khon Kaen}, \text{Bangkok}\} \cup \{3, 4\} = \{\text{Khon Kaen}, \text{Bangkok}, 3, 4\}$
- $\{1, 2\} \cup \emptyset = \{1, 2\}$



Union 2

Properties of the union operation

- $A \cup \emptyset = A$ Identity law
- $A \cup U = U$ Domination law
- $A \cup A = A$ Idempotent law
- $A \cup B = B \cup A$ Commutative law
- $A \cup (B \cup C) = (A \cup B) \cup C$ Associative law

Intersection Operation

For sets A, B, their intersection (denoted by $A \cap B$) is the set containing all elements that are simultaneously in A and ($\wedge$) in B.

Formal definition for the intersection of two sets:

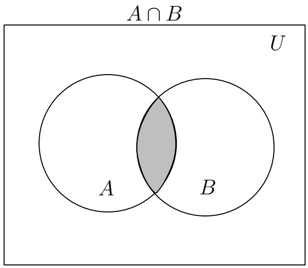
– $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Examples

- $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
- $\{\text{Khon Kaen}, \text{Bangkok}\} \cap \{3, 4\} = \emptyset$

  – No elements in common
- $\{1, 2\} \cap \emptyset = \emptyset$

  – Any set intersection with the empty set yields the empty set



Intersection

Properties of the intersection operation

- $A \cap U = A$ Identity law
- $A \cap \emptyset = \emptyset$ Domination law
- $A \cap A = A$ Idempotent law
- $A \cap B = B \cap A$ Commutative law
- $A \cap (B \cap C) = (A \cap B) \cap C$ Associative law

Disjoint Sets

Two sets are disjoint if the have no elements in common.

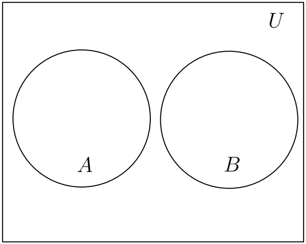
Formally, two sets A, B are called disjoint (i.e., unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)

Examples

- $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are not disjoint.
- $\{\text{Khon Kaen}, \text{Bangkok}\}$ and $\{3, 4\}$ are disjoint.
- $\{1, 2\}$ and $\emptyset$ are disjoint.

  – Their intersection is the empty set.
- What about $\emptyset$ and $\emptyset$?

  – Disjoint!. Their intersection is the empty set.



Set Difference

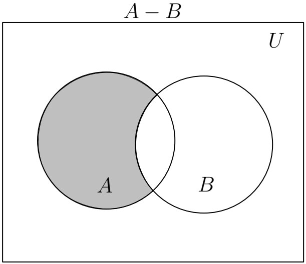
For sets A, B, the difference of A and B, written $A - B$, is the set of all elements that are in A but not B.

- $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
- $A - B = A \cap \overline{B}$ $\longleftarrow$ Important!

Examples

- $\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$
- {Khon Kaen, Bangkok} - {3, 4} = {New York, Washington}
- {1, 2} - $\emptyset$ = {1, 2}

  – The difference of any set S with the empty set will be the set S.



Set Compliment

A complement of a set is all the elements that are not in the set.

Formal definition for the complement of a set: $\overline{A} = \{x \mid x \notin A\}$.

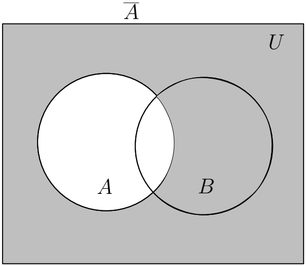
– Or $U - A$, where U is the universal set.

Further examples (assuming $U = \mathbb{Z}$)

- $\overline{\{1, 2, 3\}} = \{\ldots, -2, -1, 0, 4, 5, 6, \ldots\}$

Properties of complement sets

- $\overline{\overline{A}} = A \longleftarrow$ Complementation law
- $A \cup \overline{A} = U \longleftarrow$ Complement law
- $A \cap \overline{A} = \emptyset \longleftarrow$ Complement law



Set Identities

Set identities are basic laws on how set operations work.

- Many have already been introduced on previous slides.

Just like logical equivalences!

- Replace $\cup$ with $\vee$
- Replace $\cap$ with $\wedge$
- Replace $\emptyset$ with **F**
- Replace U with **T**

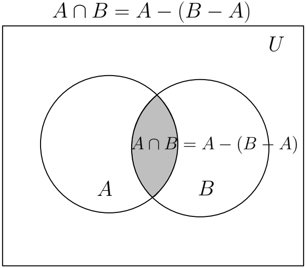
De Morgan's Laws

These should look very familiar:

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proving Set Identities

Example : $A \cap B = B - (B - A)$



Four methods can be used:

- Use the basic set identities
- Use membership tables
- Prove each set is a subset of each other

  – This is like proving that two numbers are equal by showing that each is less than or equal to the other.
- Use set builder notation and logical equivalences

Proof by Using Set Identities

Proof :

$$
\begin{aligned}
A \cap B &= B - (B - A) && \text{Statement to be proved} \\
&= B - (B \cap \overline{A}) && \text{Definition of difference} \\
&= B \cap \overline{(B \cap \overline{A})} && \text{Definition of difference} \\
&= B \cap (\overline{B} \cup \overline{\overline{A}}) && \text{De Morgan's Laws} \\
&= B \cap (\overline{B} \cup A) && \text{Complementation Law} \\
&= (B \cap \overline{B}) \cup (B \cap A) && \text{Distributive Law} \\
&= \emptyset \cup (B \cap A) && \text{Complementation Law} \\
&= B \cap A && \text{Identity Law} \\
&= A \cap B && \text{Commutative Law}
\end{aligned}
$$

Proving by Using a Membership Table

Membership tables show all the combinations of sets an element can belong to

– 1 means the element belongs, 0 means it does not

Consider the following membership table:

A	B	$A \cup B$	$A \cap B$	$A - B$
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

Proving by Using a Membership Table 2

The following membership table shows that $A \cap B = B - (B - A)$:

A	B	$A \cap B$	$B - A$	$B - (B - A)$
1	1	1	0	1
1	0	0	0	0
0	1	0	1	0
0	0	0	0	0

Because the two indicated columns have the same value, the two expressions are identical.

Note : this is similar to boolean logic.

Proof by showing each set is a subset of the other

Assume that an element is a member of one of the identities,

– **Then show it is a member of the other**

Repeat for the other identity.

We are trying to show:

- $(x \in A \cap B \rightarrow x \in B-(B-A)) \wedge (x \in B-(B-A) \rightarrow x \in A \cap B)$
- This is the biconditional:
- $x \in A \cap B \leftrightarrow x \in B-(B-A)$

Not good for long proofs.

Basically, its an English run-through of the proof.

Proof by showing each set is a subset of the other 2

Assume that $x \in B - (B - A)$

– By definition of difference, we know that $x \in B$ and $x \notin B - A$

Consider $x \notin B - A$

– If $x \in B - A$, then (by definition of difference) $x \in B$ and $x \notin A$.

– Since $x \in B - A$, then only one of the inverses has to be true (DeMorgans law): $x \notin B$ or $x \in A$

So we have that $x \in B$ **and (**$x \notin B$ **or** $x \in A$**)**

– $x \in B$ and $x \notin B$ is Universal set.

– Thus, $x \in B$ and $x \in A$

– This is the definition of intersection

Thus, if $x \in B - (B - A)$ **then** $x \in A \cap B$**.**

Proof by showing each set is a subset of the other 3

Assume that $x \in A \cap B$

– By definition of intersection, $x \in A$ and $x \in B$.

Thus, we know that $x \notin B - A$

– $B - A = \{x | x \in B \land x \notin A\}$.

Consider $B - (B - A)$

– We know that $x \notin B - A$

– We also know that if $x \in A \cap B$ then $x \in B$ (by definition of intersection)

– Thus, if $x \in B$ and $x \notin B - A$, we can restate that (using the definition of difference) as $x \in B - (B - A)$

Thus, if $x \in A \cap B$ **then** $x \in B - (B - A)$**.**

Proof by set builder notation and logical equivalences

First, translate both sides of the set identity into set builder notation.

Then massage one side (or both) to make it identical to the other

- Do this using logical equivalences

Proof by set builder notation and logical equivalences 2

$B - (B - A)$	Original statement
$= \{x \mid x \in B \wedge x \notin (B - A)\}$	Definition of difference
$= \{x \mid x \in B \wedge \neg(x \in (B - A))\}$	Negating "element of"
$= \{x \mid x \in B \wedge \neg(x \in B \wedge x \notin A)\}$	Definition of difference
$= \{x \mid x \in B \wedge (x \notin B \vee x \in A)\}$	De Morgan's Laws
$= \{x \mid (x \in B \wedge x \notin B) \vee (x \in B \wedge x \in A)\}$	Distributive Law
$= \{x \mid (x \in B \wedge \neg(x \in B) \vee (x \in B \wedge x \in A)\}$	Negating "element of"
$= \{x \mid \mathbf{F} \vee (x \in B \wedge x \in A)\}$	Negation Law
$= \{x \mid (x \in B \wedge x \in A)\}$	Identity Law
$= A \cap B$	Definition of Intersection

Proof by set builder notation and logical equivalences 3

Why can't you prove it the other way?

- That is massage $A \cap B$ to make it look like $B - (B - A)$.

You can, but its a bit annoying.

- In this case, its not simplifying the statement

Computer Representation of Set

Assume that U is finite (and reasonable!)

- Let U be the English alphabet.

Each bit represents whether the element in U is in the set

The vowels in the alphabet:

- abcdefghijklmnopqrstuvwxyz
- 10001000100000100000100000

The consonants in the alphabet:

- abcdefghijklmnopqrstuvwxyz
- 01110111011111011111011111

Computer Representation of Set 2

Consider the union of these two sets:

- 10001000100000100000100000
- $\vee$ 01110111011111011111011111
- 11111111111111111111111111

Consider the intersection of these two sets:

- 10001000100000100000100000
- $\wedge$ 01110111011111011111011111
- 00000000000000000000000000

Russell's Paradox

Most sets are not elements of themselves:

- The set of all integer $\longleftarrow$ The set is not an integer.
- The set of all horses $\longleftarrow$ The set is not a horse.

However, some sets are elements of themselves:

- The set of all the ideas.
  - Is this set an idea? Yes! So it is an element of itself.

Russel's paradox can be formally described as, let S be the set of all sets that are not elements of themselves:

$$S = \{A \mid A \text{ is a set and } A \notin A\}$$

Is S an element of itself? Neither!

- If yes, $S \in S$, then S contradict the set property that so S must not be in S.
- If no, $S \notin S$, then S satisfies the property of the set and should be an element in S.

Confuse?

The Barber Puzzle

To help you understand the paradox, the following example explain it in a closer-to-real-life situation.

The Barber Puzzle.

There is a town with a barber who shaves all the people (and only the people) who do not shave themselves.

Does the barber shave himself?

- If the barber does not shave himself, he is under the rule who will be shaved by the barber himself.
- If he shaves himself, he breaks the rule, that the barber does not shave someone who shaves himself.

To avoid this problem, we will restrict ourselves to not include sets which are subsets of themselves.

The Halting Problem

Given an algorithm X and input D, will X terminate?

- Is there an algorithm that will check if X with D will loop forever or halt?
- We will assume that the algorithm can take arbitrarily long time to execute, and take arbitrarily much storage space. The answer needed is simply halt of loop forever.

First shown by Alan Turing in 1936 that such algorithm does not exist.

- The argument used to prove is similar to that in Russell's paradox.
- Turing is often considered to be the father of modern computer science.
- Turing award was established in an honor of Alan Turing. It is widely considered to be the computing wold's equivalent to Nobel Prize.

Halting Problem 2

Why do we care if an algorithm to check the halting problem exits?

- It was one of the first algorithm that was shown to be undecidable.
- Of course, you can prove this by showing an algorithm which can decide if an algorithm and input halt or loop forever.
- But it much harder to prove that the program does not exist.
  - See page 270-271 in the text boox for a complete proof.
- It was part of the concept that explains Touring Machine, the abstract concept of any computer program used today.

Survey

I felt I understood the material in this slide set

- Very well
- With some review, Ill be good
- Not really
- Not at all

Survey 2

The pace of the lecture for this slide set was

- Fast
- About right
- A little slow
- Too slow