Lecture 2

188 200
Discrete Mathematics and Linear Algebra

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen Uiversity

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Overview

In the last lecture we studied the propositional logic which assumes that the world contains facts that are either true or false. But it doesn't solve many majority of everyday and mathematical situations.

Example: "x is greater than 3"

"All human being are immortal"

As we have learned from the last lecture, both statement are neither true or false. In this lecture we will learn how to obtain the validity from such statements. The analysis that study such logic is predicate logic (or predicate calculus).

Reference

- Chapter 2: 2.1-2.3.

Predicate Logic

Predicate, in words, refers to the part of a sentence that gives information about the subject.

Example: "x is greater than 3" and "y is a student at KKU"

Let P stand for "is greater than 3"

Let Q stand for "is a student at KKU"

Then P and Q are predicate symbols. And the sentences can be symbolized as $P(x)$ and $Q(y)$ where x and y are predicate variables.

Definition: A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that can be substituted in place of the variable.

When x is substitute with 2, $P(2)$ becomes a proposition.

Quantification

One obvious way to change predicates into proposition (recall proposition must be true or false) is to assign specific values to all their variables.

Example: Let $Q(x, y)$ denote "x is friend of y".
If you replace x with y with Tomm and Mook, $Q(\text{Tomm}, \text{Mook})$ becomes a proposition.

Another way is to add quantifier.

Quantifiers are words that refer to quantities such as "some" or "all" and tell how many elements a predicate is true.

There are two types of quantifiers:

1. Universal quantifier
2. Existential quantifier

Universal Quantifier

The **universal quantification** of $P(x)$ is the proposition "$P(x)$ is true for all values of x in the domain".

We use the notation $\forall x\ P(x)$ which can be read for all x, $P(x)$ or for every x, $P(x)$ or for arbitrary x, $P(x)$ or for any x, $P(x)$ or for each x, $P(x)$ or given any x, $P(x)$.

Example: Let $D = \{1, 2, 3, 4, 5\}$, consider the statement $\forall x \in D$, $x^2 > x$.

Solution: Since when $x = 1$ the statement is false, $\forall x \in D$, $x^2 > x$ is false.

Example: Let D be the set of real numbers in the range $[0, 1]$, consider the statement $\forall x \in D$, $x^2 \leq x$.

Solution: Since for every values of x, x^2 is no larger than x, $\forall x \in D$, $x^2 \leq x$ is true.

Existential Quantifier

The **existential quantification** of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$ is true".

We use the notation $\exists x\ P(x)$ which can be read there exists x, $P(x)$ or there is x, $P(x)$ or we can find x, $P(x)$ or there is at least one x, $P(x)$ or for some x, $P(x)$ or for at least one x, $P(x)$

Example: Let D the set of positive integers, what is the truth value of the statement $\exists x \in D$, $x^2 = x$?

Solution: Since when $x = 1$, the statement is true, so $\exists x \in D$, $x^2 = x$ is true.

Example: Let $D = \{6, 7, 8, 9, 10\}$, what is the truth value of the statement $\exists x \in D$, $x^2 = x$?

Solution: Because no matter what the value of x we substitute, the statement is always false, so $\exists x \in D$, $x^2 = x$ is false.

Summary of Quantifier Validation

So when do the universal statement and existential statement become true or false?

Statement	When true?	When false?
$\forall x\ P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false.
$\exists x\ P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Translating from Informal Language to Predicate

Example: Translate the following sentence to a predicate "all triangles have tree sides".

Solution: Let $P(x)$ denote "x has tree sides".
and T denote the set of all triangles.

$$\forall x \in T,\ P(x)$$

Example: Translating the following sentence to a predicate

" Some dogs have wings"

Solution: Let $Q(x)$ denote "x has wings".
and D denote the set of all dogs.

$$\exists x \in D,\ Q(x)$$

Translating from a Predicate to Informal Language

Example: Translate the following predicate $\forall x \in R,\ x^2 \geq 0$ where R is the set of real numbers.

Solution: There are several ways you can rewrite the predicate as informal language, for example:

- For all real numbers, the square is greater than zero.
- The square of any real number is non-negative.

Example: Translate the following predicate $\exists x \in Z,\ x^2 = x$ where Z denotes the set of all integers.

Solution:

- There is an integer whose square is equal to itself.
- For some integer x, x^2 is equal to x.

Negation of Quantified Statements

Consider the statement "All computer programers wear glasses"

What is its negation?

- Is it "no computer programmer wears glasses"?
- It actually is "some computer programmers wear glasses".

What about "Some people have no computer"?

What is its negation?

"Everyone has a computer".

Negation	Equivalent Statement	When is Negation true?	When false?
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	$P(x)$ is false for every x.	There is an x for which $P(x)$ is true.

Multiple Quantifiers

Many statements contain more than one quantifiers.

Example: For any positive number, there is another positive number that is smaller than that positive number. How do we write this as a predicate?

Solution: Let $P(x, y)$ be the statement "x is smaller than y".
D be the set of all positive numbers.

$$\forall x \in D,\ \exists y \in D,\ [(x \neq y) \wedge P(y, x)]$$

Note: Sometimes, if we know what the domain is, or the domain is obvious, we can omit it from a predicate. So $\forall x \in D,\ \exists y \in D,\ P(y, x)$ can be written as $\forall x\ \exists y,\ P(y, x)$.

Finding Truth Values of Multiple Quantified Statements

Example: Consider the statement, "for any positive integer, there is another positive integer that is smaller than that integer". What is its truth value?

If we let $P(x, y)$ denote "x is smaller than y" and the domain is all positive integers, the statement can be written as

$$\forall x \exists y, P(y, x).$$

Solution: Since when x is 1 there is no positive integer which is less than 1. Hence $\forall x \exists y, P(y, x)$ is false.

The Negation of Multiply Quantified Statements

Example: Find the negation of the following statement "Everybody loves somebody".

Solution: Let $P(x, y)$ denote "x loves y", and the domain is all the people.

Original statement can be written as

$$\forall x \exists y \; P(x, y)$$

We can use the rules for single quantifier to find its negation:

$$\begin{aligned} \neg[\forall x \exists y P(x, y)] &\equiv \neg[\forall x (\exists y P(x, y))] \\ &\equiv \exists x \neg(\exists y P(x, y)) \\ &\equiv \exists x \forall y \neg P(x, y) \end{aligned}$$

This means there is a person who love no one.

The Negation of Multiply Quantified Statements II

The negation of

$$\forall x \exists y \; P(x, y)$$

is logically equivalent to

$$\exists x \forall y, \neg P(x, y)$$

Conversely

The negation of

$$\exists x \forall y, \; P(x, y)$$

is logically equivalent to

$$\forall x \exists y, \; \neg P(x, y)$$

Properties of Quantifiers and Examples

$\forall x \forall y$ is equivalent to $\forall y \forall x$

$\exists x \exists y$ is equivalent to $\exists y \exists x$

Caution !

$\forall x \exists y$ is not equivalent to $\exists y \forall x$

Example: Find a predicate of the following statements.

Let $P(x, y)$ represent the statement " x loves y", and the domain be all the people.

Everybody loves Tom. $\longleftarrow \forall x \; P(x, Tom)$

Everybody loves somebody. $\longleftarrow \forall x \exists y \; P(x, y)$

There is somebody whom somebody loves. $\longleftarrow \exists x \exists y \; P(x, y)$

More Love Affair Examples

Nobody loves everybody.

This is equivalent to "For each person there is someone that they do not love" $\longleftarrow$ $\neg\exists x \forall y\ P(x, y) \equiv \forall x \exists y\ \neg P(x, y)$

There is somebody whom Pim doesn't love. $\longleftarrow$ $\exists y\ \neg P(Pim, y)$

There is somebody whom noone love. $\longleftarrow$ $\exists y \forall x\ \neg P(x, y)$

Everybody loves himself or herself. $\longleftarrow$ $\forall x\ P(x, x)$

There is someone who loves no one beside herself or himself. $\longleftarrow$ $\exists x \forall y\ [\neg P(x, y) \leftrightarrow (x \neq y)] \equiv \exists x \forall y\ [P(x, y) \leftrightarrow (x = y)]$

Another Example : $\exists x\ P(x) \rightarrow Q(x)$ where $P(x)$ stands for "x is a student at KKU" and "$Q(x)$ stands for x is smart".

When is this true?

It is true when there is anyone who is not a student at KKU.

More Examples

Example Everyone has exactly one best friend.

Let $P(x, y)$ stand for "y is the best friend of x".

$\forall x \exists y \; P(x, y) \longleftarrow$ **No!**

"exactly one" $\longleftarrow$ This is the problem.

$\forall x \exists y \forall z \; [P(x, y) \wedge ((z \neq y) \rightarrow \neg P(x, z))]$

or $\forall x \exists y \forall z \; [P(x, y) \wedge P(x, z) \rightarrow (z = y)]$

For everyone there is someone who is his best friend, and everyone else is not his best friend.

What about "everyone has exactly two best friends"?

More Examples

Let $Q(x, y)$ denote $x + y = 0$; consider the domain of real numbers.

What is the truth value of?

- $\exists x \forall y \; Q(x, y) \longleftarrow$ **False!**

There is one x for all y such that $Q(x, y)$ is true.

- $\forall y \exists x \; Q(x, y) \longleftarrow$ **True!**

For every x there is y (not necessary the same y), such that $Q(x, y)$ is true.

Rules of Inference for Quantified Statements

Universal Instantiation

$$\forall x\ P(x)$$
$$\therefore P(c)$$

Universal Generalization

$P(c)$ for an arbitrary c

$\therefore \forall x\ P(x)$

Existential Instantiation

$$\exists x\ P(x)$$

$\therefore P(c)$ for some element c

Existential Generalization

$P(c)$ for some element c

$\therefore \exists x\ P(x)$

Rules of Inference II

Universal Modus Ponens

$\forall x,\ P(x) \rightarrow Q(x)$

$P(c)$ for a particular c

$\therefore Q(c)$

Universal Modus Tollens

$\forall x,\ P(x) \rightarrow Q(x)$

$\neg Q(c)$, for a particular c

$\therefore \neg P(c)$

Example

Let $P(x)$ denote x is taking 188 200 class.

Let $Q(x)$ denote x has taken a course in computer engineering. (at least one class, not necessary 188 200)

Consider the premises

$\forall x\ P(x) \rightarrow Q(x)$

$P(\text{Mike})$

We can conclude $Q(\text{Mike})$

What rule of inference did we use?

Universal Modus Ponens

Argument

We will rewrite the last example in a more formal approach.

Step	**Reason**
1. $\forall x \ (P(x) \rightarrow Q(x))$	Premise
2. $P(\text{Mike}) \rightarrow Q(\text{Mike})$	Universal Instantiation
3. $P(\text{Mike})$	Premise
4. $Q(\text{Mike})$	Modus Ponens from 2 and 3

Example

Show that the premises:

1. A student in this class has not read the textbook.
2. Everybody in this class passed the midterm exam.

imply

Someone who has passed the midterm exam has not read the textbook.

Solution

Let $C(x)$ denote x is in this class.
$T(x)$ denote x has read the textbook.
$P(x)$ denote x passed the midterm exam.

Premises:

$\exists x \; (C(x) \land \neg T(x))$

$\forall x \; (C(x) \rightarrow P(x))$

We want to show that $\exists x \; (P(x) \land \neg T(x))$

Argument

Step	Reason
1. $\exists x\,(C(x) \wedge \neg T(x))$	Premise
2. $C(a) \wedge \neg T(a)$	Existential Instantiation from 1
3. $C(a)$	Simplification 2
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal Instantiation from 4
6. $P(a)$	Modus Ponens from 3 and 5
7. $\neg T(a)$	Simplification from 2
8. $P(a) \wedge \neg T(a)$	Conjunction from 6 and 7
9. $\exists x\ P(x) \wedge \neg T(x)$	Existential generalization from 8

Recap

- Predicate logic
- Quantifiers
- Multiple Quantifiers Statements.
- Negating Quantifiers
- Rules of Inference