Lecture 19

188 200
Discrete Mathematics and Linear Algebra

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen Uiversity

Overview

Topics for today.

- Subspace
- Subspace of spanned sets
- Column Space
- Null Space
- Basis for Subspace
- Coordinate system
- Dimension
- Rank

Reference : Section 2.8-2.9

Subspace of $\mathbb{R}^n$

Focus on sets of vectors in $\mathbb{R}^n$ that is called **subspaces**.

Definition: A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

1. The zero vector is in H.
2. For each $\overrightarrow{u}$ and $\overrightarrow{v}$ in H, the sum $\overrightarrow{u} + \overrightarrow{v}$ is in H.
3. For each $\overrightarrow{u}$ in H and each scalar c, $c\overrightarrow{u}$ is in H.

Note: This means a subspace is closed under addition and scalar multiplication.

Example 1: If $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are in $\mathbb{R}^n$ and $H = \text{Span}\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$, then H is a subspace of $\mathbb{R}^n$. Verify the statement.

Solution: Check if H has the three properties.

- Is $\overrightarrow{0}$ in H?
  - Yes, because $0\overrightarrow{v_1} + 0\overrightarrow{v_2}$.
- For all $\overrightarrow{u}$ and $\overrightarrow{v}$ in H, is $\overrightarrow{u} + \overrightarrow{v}$ in H?
  - For any $\overrightarrow{u}$ and $\overrightarrow{v}$ in H let $\overrightarrow{u} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2}$ and $\overrightarrow{v} = t_1\overrightarrow{v_1} + t_2\overrightarrow{v_2}$.
  - Hence we have $\overrightarrow{u} + \overrightarrow{v} = (s_1 + t_1)\overrightarrow{v_1} + (s_2 + t_2)\overrightarrow{v_2}$. So $\overrightarrow{u} + \overrightarrow{v}$ is in H.
- For any scalar c, is $c\overrightarrow{u}$ in H?
  - $c\overrightarrow{u} = c(s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2}) = (cs_1)\overrightarrow{v_1} + (cs_2)\overrightarrow{v_2}$ which is in H.

Subspace of Spaned Sets

- If $\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ are in $\mathbb{R}^n$, the set of all linear combinations of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ is a subspace of $\mathbb{R}^n$.
  - The set of all linear combination of $\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$ is $\text{Span}\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$.
  - From Example 1, $\text{Span}\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$ is a subspace of $\mathbb{R}^n$.
  - Henceforth, we shall now refer to $\text{Span}\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_2}\}$ as the **subspace spanned** (or **generated**) by $\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}$.
- $\mathbb{R}^n$ is a subspace of itself.
- The set consist of only zero vector is also a subspace in $\mathbb{R}^n$.
  - This set is called the zero subspace.

Column Space

Definition: The column space of matrix A is the set Col A of all linear combinations of the columns of A.

Example 2: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

Determine if $\overrightarrow{b}$ is in Col A.

Solution:

Note:

- Col A of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.
  - Since if $A = [\overrightarrow{a_1} \ldots \overrightarrow{a_n}]$, Col $A = \text{Span}\{\overrightarrow{a_1}, \ldots, \overrightarrow{a_n}\}$.
- Example 2 shows that Col A in $A\overrightarrow{x} = \overrightarrow{b}$ is the set of all $\overrightarrow{b}$ for which the system has a solution.

Null Space

Definition: The null space of a matrix A is the set Nul A of all solutions to $A\overrightarrow{x} = \overrightarrow{0}$.

Theorem 12: Nul A of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all the solutions to $A\overrightarrow{x} = \overrightarrow{0}$ of m homogeneous linear equations in n unknown is a subspace of $\mathbb{R}^n$.

Note:

- To check if any $\overrightarrow{v}$ is in Nul A, compute $A\overrightarrow{v}$ to see if it is $\overrightarrow{0}$.
- Nul A is defined implicitly since Nul A is defined by a condition that must be checked for each $\overrightarrow{v}$.
- Col A is defined explicitly since Col A can be constructed from the columns of A.

Example 3: Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$.

Determine if $\overrightarrow{b}$ is in Nul A.

Solution:

Basis For a Subspace

Instead of looking at the whole subspace, it is usually better to work with some subsets of subspace. We consider the smallest set of vectors that spans subspace that is called **basis**.

Definition: A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Example 4: Find a basis for $\mathbb{R}^n$.

Solution:

How to Identify a Basis

Example 5: Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution:

Finding a Basis For Col A

Example 6: Find a basis for Col A of the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

Linear Dependence Relationships

The columns of A have exactly the same linear dependence relationships as the columns of B where B is row reduced from A.

- The linear dependence relation among the columns of A can be express as the form $A\overrightarrow{x} = \overrightarrow{0}$ for some $\overrightarrow{x}$.

Example 7: It can be verified that the matrix

$$B = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

is row equivalent to the matrix A in Example 5. Find a basis for Col A.

From the previous two examples, we obtain the following theorem.

Theorem 13: The pivot columns of a matrix A form a basis for the column space of A.

Note: To use theorem 13, make sure to use the pivot columns of A itself for the basis of Col A. This implies that you should not use the pivot columns of an echelon form matrix of A since they are not often in Col A.

Nul A VS Col A

1. Nul A is a subspace of $\mathbb{R}^n$
2. Nul A is implcitly defined.
3. Take time to find vectors in Nul A. Row operations on $[\ A \quad \overrightarrow{0}\ ]$ are required.
4. There is no obvious relation between Nul A and the entries in A.
5. A typical vector $\overrightarrow{v}$ in Nul A has the property that $A\overrightarrow{v} = \overrightarrow{0}$.
6. Easy to check if $\overrightarrow{v}$ is in Nul A. Compute $A\overrightarrow{v}$.

1. Col A is a subspace of $\mathbb{R}^n$.
2. Col A is explicitly defined.
3. Easy to find. The columns of A are displayed; others are formed from them.
4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\overrightarrow{v}$ in Col A has the property that $A\overrightarrow{x} = \overrightarrow{v}$ is consistent.
6. Harder to check, compute $[\ A \quad \overrightarrow{v}\ ]$.

Nul A VS Col A

7. Nul $A = \{\overrightarrow{0}\}$ iff $A\overrightarrow{x} = \overrightarrow{0}$ has only trivial solution.
8. Nul $A = \{\overrightarrow{0}\}$ iff $x \mapsto Ax$ is one-to-one.

7. Col $A = \mathbb{R}^m$ iff $A\overrightarrow{x} = \overrightarrow{b}$ has a solution for every $\overrightarrow{b}$.
8. Col A $= \mathbb{R}^m$ iff $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Coordinate Systems

We select a basis instead of subspace for H because each vector in H can be written in only one way as a linear combination of the basis.

Definition: Suppose $\mathcal{B} = \{\overrightarrow{b_1}, \dots, \overrightarrow{b_p}\}$ is a basis of a subspace H. For each $\overrightarrow{x}$ in H, the coordinates of $\overrightarrow{x}$ relative to $\mathcal{B}$ are the weight $c_1, \dots, c_p$ s.t. $\overrightarrow{x} = c_1\overrightarrow{b_1} + \dots + c_p\overrightarrow{b_p}$, and the vector in $\mathbb{R}^p$

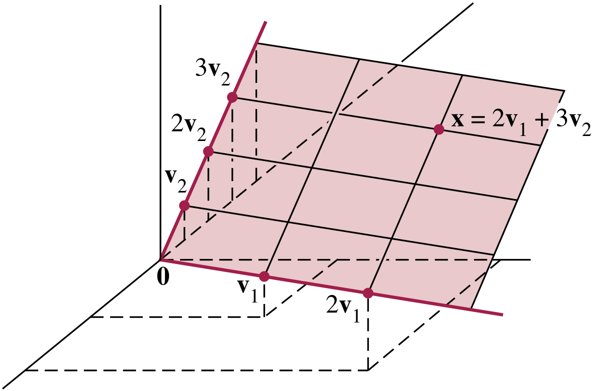
$$[\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the coordinate vector of $\overrightarrow{x}$ (relative to $\mathcal{B}$) or the $\mathcal{B}$-coordinate vector of $\overrightarrow{x}$.

Example 8: Let $\overrightarrow{v_1} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\overrightarrow{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\overrightarrow{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ and $\mathcal{B} = \{\overrightarrow{v_1}, \overrightarrow{v_2}\}$. Then $\mathcal{B}$ is a basis for H=Span$\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ because $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are linearly independent. Determine if $\overrightarrow{x}$ is in H, and if it is, find the coordinate vector of $\overrightarrow{x}$ relative to $\mathcal{B}$.

Solution:

The figure shows a coordinate system on a plane H in $\mathbb{R}^3$



- Notice although H is in $\mathbb{R}^3$ they are completely determined by their coordinate which are in $\mathbb{R}^2$.
- The grid on the plane makes H looks like $\mathbb{R}^2$.

- The correspondence $\overrightarrow{x} \mapsto [\overrightarrow{x}]_{\mathcal{B}}$ is a one-to-one correspondence between H and $\mathbb{R}^2$ that preserve linear combination.
- We call such a correspondence an isomorphism, and we say H is isomorphic to $\mathbb{R}^2$.
- In general, if $B = \{\overrightarrow{b_1}, \ldots, \overrightarrow{b_p}\}$ is a basis for H, then $\overrightarrow{x} \mapsto [\overrightarrow{x}]_{\mathcal{B}}$ is a one-to-one correspondence that makes H look and act the same as $\mathbb{R}^p$, even though H may have more than p entries.

The Dimension of a Subspace

Definition: The **dimension** of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\vec{0}\}$ is defined to be zero.

Note:

- The zero subspace has no basis because $\{\vec{0}\}$ is a linearly dependent set.
- The space $\mathbb{R}^n$ has dimension $n \longrightarrow$ every basis for $\mathbb{R}^n$ has n vectors.
- A plane through $\vec{0}$ in $\mathbb{R}^3$ is two dimensional.
- A line through $\vec{0}$ is one dimensional.

Rank of a Matrix

Definition: The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

Note: Recall that the pivot columns of A from a basis for Col A.

Example 9: Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

Solution:

Theorems

Theorem: (**The Rank Theorem**) If a matrix A has n columns, then rank A + dim Nul $A = \mathrm{n}$.

Theorem (**The Basis Theorem**) Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

IMT (continued)

Theorem: The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

a. A is an invertible matrix.

$\vdots \qquad\qquad \vdots$

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\overrightarrow{0}\}$

r. $\dim \text{Nul } A = 0$

Recap

- Subspace
- Subspace of spanned sets
- Column Space
- Null Space
- Basis for Subspace
- Coordinate system
- Dimension
- Rank

Next time, we will start start Chapter 5 with Eigenvectors and Eigenvalues.