Lecture 18

188 200
Discrete Mathematics and Linear Algebra

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Overview

Topics for today.

- Partitioned matrices
- Addition
- Multiplication
- Inverse
- Matrix factorization
- LU factorization
- The LU factorization algorithm

Reference : Section 2.4-2.5

Partitioned Matrices

- Thus far, a key feature working with matrices is the ability to partition a matrix into a list of column vectors.
- We wish to consider other partitions of a matrix.

Example 1: Consider the 3×6 matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

This can also be written as 2×3 partitioned (or block) matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the block (or submatrices) A_{11}, A_{12}, A_{13}, A_{21}, A_{22} and A_{23} corresponding to the original matrix A.

Addition and Multiplication of Partitioned Matrices

- If A and B are the same size and are partitioned the same way, $A+B$ is defined.

  • Each block of $A+B$ is the sum of the corresponding block of A and B.

  Exmaple 2: Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

$$A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}$$

- If the column partition of A matches the row partition of B, AB is defined.
  - AB can be computed normally as if blocks are scalar.

Multiplication of Partitioned Matrices, Example

Example 3: Let

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

AB is defined and equals to

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

Inverses of Partitioned Matrices

The next example illustrate how to find an inverse of a partitioned matrix.

Example 4: Consider the matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible. Find a formula for A^{-1}.

Solution: Denote A^{-1} by B and partition B so that

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

This matrix provide the following four equations:

$$A_{11}B_{11} + A_{12}B_{21} = I_p \quad (1)$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \quad (2)$$

$$A_{22}B_{21} = 0 \quad (3)$$

$$A_{22}B_{22} = I_p \quad (4)$$

Example 4, cont.

- Since A is a square matrix, (4) implies that A_{22} is invertible.
  - So $B_{22} = A_{22}^{-1}$
- (3) and the fact that A_{22} is invertible means $B_{21} = A_{22}^{-1}0 = 0$
- So (1) can be simplified to $A_{11}B_{11} + 0 = I_p$.
  - So $B_{11} = A_{11}^{-1}$.
- From (2), $A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1}$
  - So $B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$.

Combining everything together we have

$$B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

Matrix Factorization

- **A factorization** of a matrix A is an equation that express A as two or more matrices.
- The motivation to find a factorization of a matrix came from the problem of solving a sequence of equations, all with the same coefficient matrix.
  - $A\overrightarrow{x} = \overrightarrow{b_1}$, $A\overrightarrow{x} = \overrightarrow{b_2}$, ..., $A\overrightarrow{x} = \overrightarrow{b_p}$.
- This can be done, if A is invertible, by compute $A^{-1}\overrightarrow{b_1}$, $A^{-1}\overrightarrow{b_2}$, ..., $A^{-1}\overrightarrow{b_p}$.
- Instead of repeatedly compute multiplication of A^{-1} with all the vectors $\overrightarrow{b}$, we will use a factorization of a matrix called **LU factorization** which will speed up the calculation.

LU factorization

Suppose A is an $m \times n$ invertible matrix that can be row reduced to echelon form without row interchange. Then A can be written as $A = LU$.

- L is an $m \times m$ **lower triangular matrix** with 1's on the diagonal. L is invertible and is called a unit lower triangular matrix. Example of L

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

- U is an $m \times n$ echelon form of A.
- When $A = LU$ then $A\overrightarrow{x} = \overrightarrow{b} \rightarrow L(U\overrightarrow{x}) = \overrightarrow{b}$.
  - So to find $\overrightarrow{x}$, we solve $L\overrightarrow{y} = \overrightarrow{b}$ and $U\overrightarrow{x} = \overrightarrow{y}$.
  - This is easy to solve since L and U are triangle.

Is it really easier or faster?

Before learn how to find L and U, the next example shows that LU factorization is useful.

Example 5: Let $A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$ and let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}, \ U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \ \overrightarrow{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}.$$

It can be verified that $A = LU$. Solve $A\overrightarrow{x} = \overrightarrow{b}$.

- First solve $L\overrightarrow{y} = \overrightarrow{b}$.
  - Solve for $\overrightarrow{y}$ by row reducing $\begin{bmatrix} L & \overrightarrow{b} \end{bmatrix}$ to $\begin{bmatrix} I & \overrightarrow{y} \end{bmatrix}$

$$\begin{bmatrix} L & \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & \overrightarrow{y} \end{bmatrix}$$

- This process uses 6 multiplications and 6 additions.
- Next solve $U\overrightarrow{x} = \overrightarrow{y}$. $\begin{bmatrix} U & \overrightarrow{y} \end{bmatrix} =$

$$\begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

- This requires 4 divisions, 6 multiplications and 6 additions.
- In total, we only needs **28 arithmetic operations**.
- How many operations to row reduce $\begin{bmatrix} A & \overrightarrow{b} \end{bmatrix}$ to $\begin{bmatrix} I & \overrightarrow{x} \end{bmatrix}$? 62

An LU Factorization Algorithm

The efficiency of LU factorization depends on knowing L and U.
The next algorithm show how to efficiently obtained L and U.

LU **factorization algorithm**:

1. Reduce A to an echelon form U by a sequence of row replacement operations if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Note:

- Step 1 is not always possible. If it is, it implies that LU factorization exist.
- Make sure that when reduce A to U only row replacement operations are used.
- Using LU factorization to solve a sequence of equations, L and U are computed just once.

LU Factorization Algorithm, Proof

Proof:

- Suppose A can be row reduced to echelon form U using only row replacement.
- There exist a unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that $E_p \cdots E_1 A = U$.
  - Example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$
- Then $A = (E_p \ldots E_1)^{-1} U = LU$ where $L = (E_p \ldots E_1)^{-1}$.
- Since products and inverses of unit lower triangular matrices are also lower triangular, L is unit lower triangular.
- The row operations that reduce A to U, will also reduce L to I, because $E_p \ldots E_1 L = (E_p \ldots E_1)(E_p \ldots E_1)^{-1}$.

Example 6: Find L and U of the following matrix

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Solution:

Example 7: ind L and U of the following matrix

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

Solution:

Recap

- Partitioned matrices
- LU factorization

Next time, we will close the third chapter with Subspaces, Dimension and Rank.