Lecture 17

188 200
Discrete Mathematics and Linear Algebra

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Overview

Our ability to analyze and solve equations will be greatly enhances when we can perform algebraic operations with matrices. For the next couple of lectures we will learn some basic tools for handling two or more matrices.

Topics for today.

- Matrix notations
- Scalar multiplication and Matrix sum
- Matrix multiplication
- The transpose of matrix
- The inverse of matrix
- The algorithm to find A^{-1}
- Invertible linear transformation

Reference : Section 2.1-2.3

Matrix Notation

Two ways to denote $m \times n$ matrix A:

1. In terms of the columns of A:

$$A = \begin{bmatrix} \overrightarrow{a_1} & \overrightarrow{a_2} & \dots & \overrightarrow{a_n} \end{bmatrix}$$

2. In terms of the entries of A:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Terminologies in Matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Recall: a_{ij} is the entry at i-th column and j-th row.

- **Diagonal entries** are $a_{11}, a_{22}, a_{33}, \dots$.
- **Main diagonal**: $a_{11}, a_{22}, a_{33}, \dots$ form the main diagonal.
- **Diagonal matrix**: a square matrix whose non-diagonal entries are zero.
- **Zero matrix**: A matrix with all zero entries.

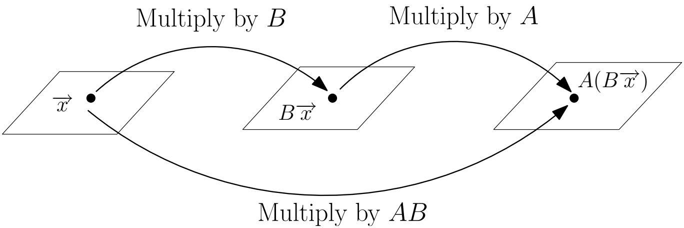
Scalars Multiple and Matrix Sum: Properties

Theorem 1: Let A, B and C be matrices of the same size, and let r and s be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Matrix Multiplication

- Multiplying B and $\overrightarrow{x}$ transforms $\overrightarrow{x}$ into $B\overrightarrow{x}$.
- Then multiplying A and $B\overrightarrow{x}$, we transform $B\overrightarrow{x}$ into $A(B\overrightarrow{x})$.
- So $A(B\overrightarrow{x})$ is the composition of two mappings.



If we denote $B = \begin{bmatrix} \overrightarrow{b_1} & \overrightarrow{b_2} & \dots & \overrightarrow{b_p} \end{bmatrix}$ and $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ then

$$B\overrightarrow{x} = x_1\overrightarrow{b_1} + \dots + x_p\overrightarrow{b_p}.$$

Matrix Multiplication, cont.

By the linearity of multiplication by A

$$\begin{aligned} A(B\overrightarrow{x}) &= A(x_1\overrightarrow{b_1} + \ldots + x_p\overrightarrow{b_p}) \\ &= x_1A\overrightarrow{b_1} + x_1A\overrightarrow{b_2} + \ldots + x_1A\overrightarrow{b_p} \\ &= \begin{bmatrix} A\overrightarrow{b_1} & \ldots & A\ \overrightarrow{b_p} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \end{aligned}$$

Hence we have,

Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix whith columns $\overrightarrow{b_1}, \ldots, \overrightarrow{b_p}$, then the product AB is the $m \times p$ matrix whose columns are $A\overrightarrow{b_1}, \ldots, A\overrightarrow{b_p}$. That is,

$$AB = A\begin{bmatrix} \overrightarrow{b_1} & \overrightarrow{b_2} & \ldots & \overrightarrow{b_p} \end{bmatrix} = \begin{bmatrix} A\overrightarrow{b_1} & A\overrightarrow{b_2} & \ldots & A\overrightarrow{b_p} \end{bmatrix}$$

Matrix Multiplication, Example (first method)

Example 1: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

Note: $A\overrightarrow{b_1}$ is a linear combination of the columns of A and $A\overrightarrow{b_2}$ is a linear combination of the columns of A.

Observation: Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

Size of Multiplication Matrix

Example 2: If A is 4×3 and B is 3×2, then what are the size of AB and BA?

Note: If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Row-Column Rule for Computing AB (second method)

If the product AB is defined then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If AB_{ij} denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$AB_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 3: $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB and BA, if they are defined.

Solution:

Properties of Matrix Multiplication

Theorem 2: Let A an $m \times n$ matrix, and let B and C have size for which the indicated sums and products are defined.

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B + C) = AB + AC$ (left distributive law)
3. $(B + C)A = BA + CA$ (right distributive law)
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m = A = AI_n$ (identity for matrix multiplication)

Warning:

1. In general $AB \neq BA$
   - If $AB = BA$, we say that A and B **commute** with one another.
2. Even if $AB = AC$, then B may not equal to C.
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$.

Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of copies of A:

$$A^k = \underbrace{A \dots A}_{k}$$

Note: A^0 is defined as an identity matrix.

The Transpose of a Matrix

Definition: Given $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denotes A^T, whose columns are formed from the corresponding rows of A.

Example 4: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$$

What are A^T and B^T?

Properties of Transpose Matrices

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

1. $(A^T)^T = A$ (I.e., the transpose of A^T is A)
2. $(A + B)^T = A^T + B^T$
3. For any scalar r, $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the inverse of A.

Fact: If A is invertible, then the inverse is unique.

The inverse of A is usually denoted by A^{-1}.

We have

$$AA^{-1} = A^{-1}A = I_n$$

Note: Not all $n \times n$ matrices are invertible. A matrix which is not invertible is sometimes called a singular matrix. An invertible matrix is called nonsingular matrix.

An Inverse of a 2×2 matrix

Theorem 4: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Theorem 5: If A is an invertible $n \times n$ matrix, then for each $\overrightarrow{b}$ in $\mathbb{R}^n$, the equation $A\overrightarrow{x} = \overrightarrow{b}$ has the unique solution $\overrightarrow{x} = A^{-1}\overrightarrow{b}$.

Example 5: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ -5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

Solution:

Theorem 6: Suppose A and B are invertible. Then the following results hold:

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1})
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Earlier we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary matrices**.

Elementary Matrices

Definition: An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Example 6: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,
$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ E_1, E_2 are E_3 are elementary matrices. Why?

Example 6, cont.

Example 7: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $E_3 E_2 E_1 = I_3$.

Then multiplying on the right by A^{-1}, we get $E_3 E_2 E_1 I_3 = A^{-1}$

Algorithm for Finding The Inverse of A

The elementary row operations that row reduced A to I_n are the same elementary row operations that transform I_n into A^{-1}.

Theorem 7: An $n \times n$ matrix A is invertible iff A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduce A to I_n will also transform I_n to A^{-1}.

Algorithm for finding A^{-1}

1. Place A and I side-by-side to form an augmented matrix $[A \quad I]$.
2. Perform row operations on this matrix (which will produce identical operations on A and I).
3. So by Theorem 7:

$$[A \quad I] \text{ will row reduce to } \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

or A is not invertible.

Example 8: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ if it exists.

Solution:

Example 9: Find the inverse of $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$ if it exists.

Solution:

Characterizations of Invertible Matrices

Theorem 8 : The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statement are equivalent.

a. A is an invertible matrix.
b. A is row equivalent to I_n.
c. A has n pivot positions.
d. The equation $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\overrightarrow{x} \mapsto A\overrightarrow{x}$ is one-to-one.
g. The equation $A\overrightarrow{x} = \overrightarrow{b}$ has at least one solution for each $\overrightarrow{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.

Theorem 8 (cont.)

i. The linear transformation $\overrightarrow{x} \mapsto A\overrightarrow{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I_n$.

l. A^T is an invertible matrix.

Example 11: Use the Invertible Theorem Matrix to determine if A is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}$$

Solution:

Example 12: Suppose H is 5×5 and suppose there is a vector v in $\mathbb{R}^5$ which is not a linear combination of the columns of H. What can you say about the number of solutions to $H\overrightarrow{x} = \overrightarrow{0}$?

Solution:

Invertible Linear Transformations

For an invertible matrix A,

- $A^{-1}A\overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^n$, and
- $AA^{-1}\overrightarrow{x} = \overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^n$.

Pictures:

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

- $S(T(\overrightarrow{x})) = \overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^n$, and
- $T(S(\overrightarrow{x})) = \overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^n$.

Theorem 9: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible iff A is an invertible matrix. In that case, the linear transformation S given by $S(\overrightarrow{x}) = A^{?1}\overrightarrow{x}$ is the unique function satisfying

- $S(T(\overrightarrow{x})) = \overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^n$, and
- $T(S(\overrightarrow{x})) = \overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^n$.

Recap

- Matrix notations
- Scalar multiplication and Matrix sum
- Matrix multiplication
- The transpose of matrix
- The inverse of matrix
- The algorithm to find A^{-1}
- Invertible linear transformation

Next time, we see some partition matrices and matrix factorization