Lecture 16

188 200
Discrete Mathematics and Linear Algebra

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Overview

Topics for today.

- A transformation
- Matrix transformation
- Check weather a transform is linear
- The matrix of a linear transformation
- Function properties of linear transformation

Reference : Section 1.8-1.9

A Transformation

$$A\overrightarrow{x} = \overrightarrow{b} \qquad \text{VS} \qquad x_1\overrightarrow{a_1} + \ldots + x_n\overrightarrow{a_n} = \overrightarrow{b}$$

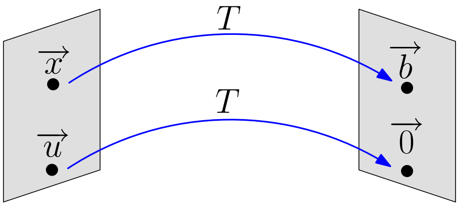
- They are basically the same.
- $A\overrightarrow{x} = \overrightarrow{b}$ is more useful, e.g. for computer graphic and signal processing.
- You can think of A as a tool to transform $\overrightarrow{x}$ to $\overrightarrow{b}$.

Example:

$$\underset{A}{\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}} \underset{\overrightarrow{x}}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \underset{\overrightarrow{b}}{\begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}} \qquad \underset{A}{\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}} \underset{\overrightarrow{u}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \underset{\overrightarrow{0}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

These show that the multiplication by A transform $\overrightarrow{x}$ into $\overrightarrow{b}$ and $\overrightarrow{u}$ into $\overrightarrow{0}$ respectively.

Graphical Representation of Transformation



With this new perspective, solving $A\overrightarrow{x} = \overrightarrow{b}$ is to find all $\overrightarrow{x}$ in $\mathbb{R}^2$ that can be transformed to $\overrightarrow{b}$ in $\mathbb{R}^3$ through multiplication by A.

Definition: A **transformation** (or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that is an assignment to each $\overrightarrow{x}$ in $\mathbb{R}^n$ $T(\overrightarrow{x})$ in $\mathbb{R}^m$. $\mathbb{R}^n$ is the **domain** of T and $\mathbb{R}^m$ is the **codomain** of T. For each $\overrightarrow{x}$, $T(\overrightarrow{x})$ is an **image** of $\overrightarrow{x}$ and the set of all images is the **range** of T.

Matrix Transformations

- Focus on mapping with matrix multiplication A.
- Sometimes denote a matrix transformation by $\overrightarrow{x} \mapsto A\overrightarrow{x}$.

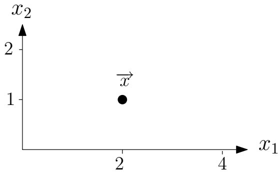
Example 1: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\overrightarrow{x}) = A\overrightarrow{x}$.

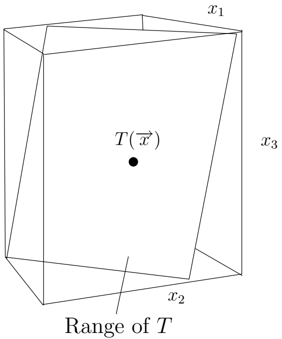
Then if $\overrightarrow{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$T(\overrightarrow{x}) = A\overrightarrow{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Example 1, cont.

$$T(\overrightarrow{x}) = A\overrightarrow{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$





Domain : $\mathbb{R}^2$

Codomain : $\mathbb{R}^3$

Example 2

Example 2: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\overrightarrow{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\overrightarrow{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\overrightarrow{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Define a transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\overrightarrow{x}) = A(\overrightarrow{x})$.

1. Find an $\overrightarrow{x}$ in $\mathbb{R}^3$ whose image under T is $\overrightarrow{b}$.
2. Is there more than one $\overrightarrow{x}$ under T whose image is $\overrightarrow{b}$ (uniqueness problem)
3. Determine $\overrightarrow{c}$ is in the range of the transformation T. (existence problem)

Solution:

Example 2, cont.

Solution: (cont.)

Is a Transformation Linear?

Definition: A transformation T is **linear** if:

1. $T(\overrightarrow{u} + \overrightarrow{v}) = T(\overrightarrow{u}) + T(\overrightarrow{v})$ for all $\overrightarrow{u}$, $\overrightarrow{v}$ in the domain of T.
2. $T(c\overrightarrow{u}) = cT(\overrightarrow{u})$ for all $\overrightarrow{u}$ and all scalars c.

Fact: Every matrix transformation is a linear transformation.

Corollary:

3. If $T(\overrightarrow{0}) = \overrightarrow{0}$
4. $T(c\overrightarrow{u} + d\overrightarrow{v}) + cT(\overrightarrow{u}) + dT(\overrightarrow{v})$

   for all $\overrightarrow{u}$, $\overrightarrow{v}$ in the domain of T and all scalars c and d.

(4) can be generalized to (known as superposition principle)

$$T(c_1\overrightarrow{v_1} + \ldots + c_p\overrightarrow{v_p}) = c_1T(\overrightarrow{v_1}) + \ldots + c_pT(\overrightarrow{v_p})$$

Example 3

Example 3: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\overrightarrow{x}) = r\overrightarrow{x}$ where r is a scalar. Show that T is a linear transformation.

Note: T is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$.

Solution:

Example 4

Example 4: Let $\overrightarrow{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\overrightarrow{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\overrightarrow{y_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\overrightarrow{y_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation which maps $\overrightarrow{e_1}$ into $\overrightarrow{y_1}$ and $\overrightarrow{e_2}$ into $\overrightarrow{y_2}$. Find the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution:

Example 4, cont.

Solution to Example 4 (cont.)

Example 5

Example 5: Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}.$$

Show that T is not a linear transformation.

Solution :

The Matrix of a Linear Transformation

- When talk about T, usually want to find a "formula" for $T(\overrightarrow{x})$.
- This can be done by observing what T does to the columns of the $n \times n$ identity matrix.

Definition: An **identity matrix** (denoted as I_n) is an $n \times n$ matrix with 1's on the left-to-right diagonal and 0's elsewhere. The i-th column of I_n is labeled $\overrightarrow{e_i}$.

Example 6: Suppose that T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\overrightarrow{e_1}) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \text{ and } T(\overrightarrow{e_2}) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Compute $T(\overrightarrow{x})$ for any $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Example 6, Solution

Solution:

Standard Matrix

Theorem 10: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\overrightarrow{x}) = A\overrightarrow{x} \text{ for all } \overrightarrow{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose j-th column is the vector $T(\overrightarrow{e}_j)$ is the j-th column of the identity matrix in $\mathbb{R}^n$:

$$A = [\ T(\overrightarrow{e_1}) \quad \dots \quad T(\overrightarrow{e_n})\ ]$$

Note: A is called standard matrix for the linear transformation T.

Example 7

Example 7: Let A be a 3×2 matrix. Find A from the following equation:

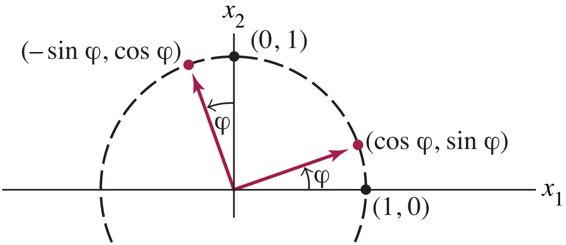
$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

Solution:

Rotation Transformation

Example 8: Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates a point about the origin through an angle of φ with counterclockwise rotation for a positive angle.

Solution:



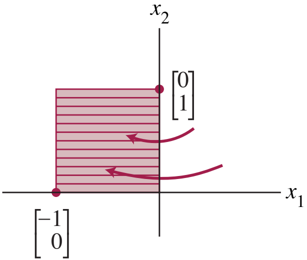
Example 8, cont.

Solution to Example 8: (cont.)

Geometric Linear Transformations of $\mathbb{R}^2$

Example 9 Find the standard matrix of the following transformation, if the input is the unit square.

Reflection in the x_2-axis

Image of the Unit Square	Standard Matrix
	$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Note: For more standard matrices check Table 1-4 in section 1.9.

Function Properties of Linear Transformation

Definition: $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\overrightarrow{b}$ in $\mathbb{R}^m$ is the image of at least one $\overrightarrow{x}$ in $\mathbb{R}^n$.

- Equivalently, T is onto if the range of $T =$ the codomain $\mathbb{R}^m$.
- This is an existence question, since T is not onto if $\exists \overrightarrow{b}$ for which $T(X) = \overrightarrow{b}$ has no solution.

Definition: $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\overrightarrow{b}$ in $\mathbb{R}^m$ is the image of at most one $\overrightarrow{x}$ in $\mathbb{R}^n$.

- Equivalently, T is one-to-one if $\forall \overrightarrow{b}$ either has a unique solution or none at all.
- This is a uniqueness question, since T is not one-to-one if $\exists \overrightarrow{b}$ that is the image of more than one $\overrightarrow{x}$ in T.

Determine if Transformation is Onto/One-to-one

Theorem 11: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff $T(\overrightarrow{x}) = \overrightarrow{0}$ has **only the trivial solution**.

Theorem 12: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and A be the standard matrix for T. Then

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$.

b. T is one-to-one iff the columns of A are linearly independent.

Example 10: Let A be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \end{bmatrix}$$

Does T maps $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Solution:

- A is in echelon form that has pivot in every rows.
  - This implies that for each $\overrightarrow{b}$ in $\mathbb{R}^3$, $A\overrightarrow{x} = \overrightarrow{b}$ has a solution.
  - By Theorem 4 in Section 1.4, this implies that T maps $\mathbb{R}^4$ onto $\mathbb{R}^3$.
- Since A has a free variable, each $\overrightarrow{b}$ is the image of more than one $\overrightarrow{x}$.
  - T is not one-to-one.

Example 11: Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.

Solution: Given that

$$T(\overrightarrow{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The columns of standard matrix A are linearly independent since they are not multiple of each other.
  - By Theorem 12b, T is one-to-one.
- T maps onto $\mathbb{R}^3$ if the columns of A span $\mathbb{R}^3$.
  - Which is true iff A has a pivot in every rows. This is impossible, hence T is not onto $\mathbb{R}^3$.

Recap

- A transformation
- Matrix transformation
- Check weather a transform is linear
- The matrix of a linear transformation
- Function properties of linear transformation

Next time, we will start Chapter 2, matrix algebra.