Lecture 15

188 200
Discrete Mathematics and Linear Algebra

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen Uiversity

Overview

Topics for today.

- Solution sets of linear systems
- Homogeneous linear systems
- Nonhomogeneous linear systems
- Parametric vector form
- Linear independence
- Linear dependence
- Special cases

Reference : Section 1.5, 1.7

Solution Sets of Linear Systems

Homogeneous Linear Systems: a linear system is said to be homogeneous if it can be written in the form

$$A\overrightarrow{x} = \overrightarrow{0}$$

where A is $m \times n$ matrix and $\overrightarrow{0}$ is the zero vector in $\mathbb{R}^m$.

Example 1: The following linear system is homogeneous.

$$\begin{aligned} x_1 + 10x_2 &= 0 \\ 2x_1 + 20x_2 &= 0 \end{aligned}$$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution Sets of Homogeneous Systems

Homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$ **always** has at least one solution, namely $\overrightarrow{x} = \overrightarrow{0}$. We call this **trivial solution**.

Example 2: From Example 1:

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\overrightarrow{x} = \overrightarrow{0}$ the equation is true.

The important question is whether there exist a **nontrivial solution**, i.e. a nonzero vector $\overrightarrow{x}$ that satisfies $A\overrightarrow{x} = \overrightarrow{0}$.

Observation: The homogeneous equation has a nontrivial solution iff the equation has at least one free variable.

Nontrivial Solution, example

Example 3: Continue from Example 2, do nontrivial solutions exist in the following equation?

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution: We can row reduce the corresponding augmented matrix to echelon form:

$$\begin{bmatrix} 1 & 10 & 0 \\ 2 & 20 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, x_2 is free variable, which implies that the system has infinitely many solutions, which also implies that nontrivial solutions exist.

Solution Sets for Homogeneous System, Example

Example 4: Determine if the following homogeneous system has nontrivial solution and then describe the solution set.

$$2x_1 + 4x_2 - 6x_3 = 0$$
$$4x_1 + 8x_2 - 10x_3 = 0$$

Solution:

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & 10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- $x_1 = -2x_2$
- x_2 is free
- $x_3 = 0$

Example 4, cont.

Recall that $x_1 = -2x_2$, x_2 is free and $x_3 = 0$. As a vector the general solution of $A\overrightarrow{x} = \overrightarrow{0}$ has the form

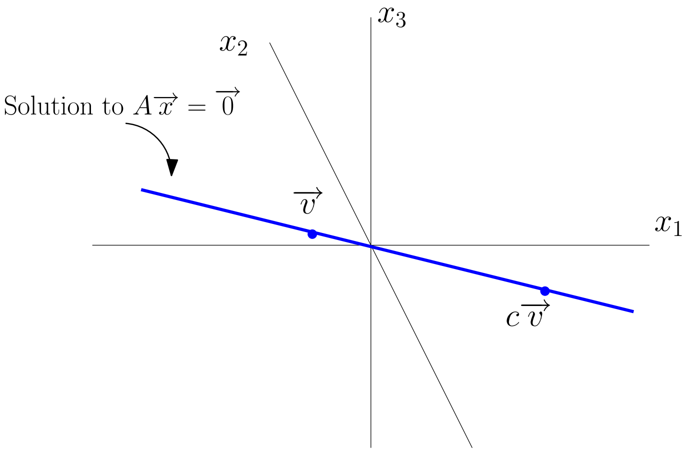
$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2 \overrightarrow{v}$$

where $\overrightarrow{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. This shows that every solutions is a scalar multiple of $\overrightarrow{v}$. And the trivial solution is when $x_2 = 0$.

Note: Actually, by looking at the linear system you can tell there is at least one free variable.

- why?

Graphical Representation



Solution set = span$\{\overrightarrow{v}\}$ = line through $\overrightarrow{0}$ in $\mathbb{R}^3$

Homogeneous System, A Single Linear Equation

Example 5: A single linear equation can be treated as a very simple system of homogeneous equation. Describe all solution of the following homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0$$

Solution:

- x_1 is basic variable. x_2 and x_3 are free variable.
- The general solution is $x_1 = 0.3x_2 + 0.2x_3$, with x_2 and x_3 are free. The general solution in term of vector is

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} \text{ with } x_2,\ x_3 \text{ free}$$

Homogeneous System , Example cont.

$$\overrightarrow{x} = x_2 \underset{\overset{\uparrow}{\overrightarrow{u}}}{\begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{\overset{\uparrow}{\overrightarrow{v}}}{\begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}}$$

- This shows that every solutions is a linear combination of $\overrightarrow{u}$ and $\overrightarrow{v}$.
- This means that the solution set is Span$\{\overrightarrow{u}, \overrightarrow{v}\}$.
- Graphically, the solution set is a plane through the origin.

Parametric Vector Form

From the previous example,

$$10x_1 - 3x_2 - 2x_3 = 0$$

is an implicit description of the plane.

The explicit description of the plane is the set spanned by $\overrightarrow{u}$ and $\overrightarrow{v}$.

$$\overrightarrow{x} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

is called parametric vector equation. It has a general form as

$$\overrightarrow{x} = s\overrightarrow{u} + t\overrightarrow{v} \qquad \text{where } s, t \text{ in } \mathbb{R}$$

Nonhomogeneous System

When a **nonhomogeneous system** has many solutions, the general solutions can be written in parametric vector form as one vector plus arbitrary linear combination of vector that satisfying homogeneous system.

Example 6: Describe all solutions of (the LHS is the same as Example 4)

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 8x_2 - 10x_3 &= 4 \end{aligned}$$

Solution:

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Exam 6, cont.

$$\begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The general solution is

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 \\ x_2 \\ 2 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \overrightarrow{p} + x_2 v$$

where $\overrightarrow{p}$ is $\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$.

Recap of Example 4 and 6

Solution $A\overrightarrow{x} = \overrightarrow{0}$

$$\overrightarrow{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2 \overrightarrow{v}$$

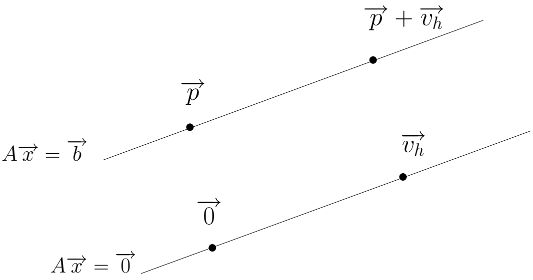
$\overrightarrow{x} = x_2 \overrightarrow{v} =$ parametric equation of line passing through $\overrightarrow{0}$ and $\overrightarrow{v}$.

Solution $A\overrightarrow{x} = \overrightarrow{b}$

$$\overrightarrow{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \overrightarrow{p} + x_2 v$$

$\overrightarrow{x} = \overrightarrow{p} + x_2 \overrightarrow{v} =$ parametric equation of line passing through $\overrightarrow{p}$ and $\overrightarrow{v}$.

Summarizing Example 4 and 6



Theorem 6 Suppose $A\overrightarrow{x} = \overrightarrow{b}$ is consistent for some given $\overrightarrow{b}$, and let $\overrightarrow{p}$ be a solution. Then the solution set of $A\overrightarrow{x} = \overrightarrow{b}$ is the set of all vectors of the form $\overrightarrow{w} = \overrightarrow{p} + \overrightarrow{v}_h$ where $\overrightarrow{v}_h$ is any solution of the homogeneous equation of $A\overrightarrow{x} = \overrightarrow{0}$.

Example 7

Example 7 Describe the solution set of $2x_1 - 4x_2 - 4x_3 = 0$ compare it to the solution set of $2x_1 - 4x_2 - 4x_3 = 6$.

Solution: Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 0$:

$$\begin{bmatrix} 2 & -4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 & 0 \end{bmatrix}$$

Vector form of the solution:

$$\overrightarrow{v_1} = \begin{bmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Example 7, cont.

Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 6$:

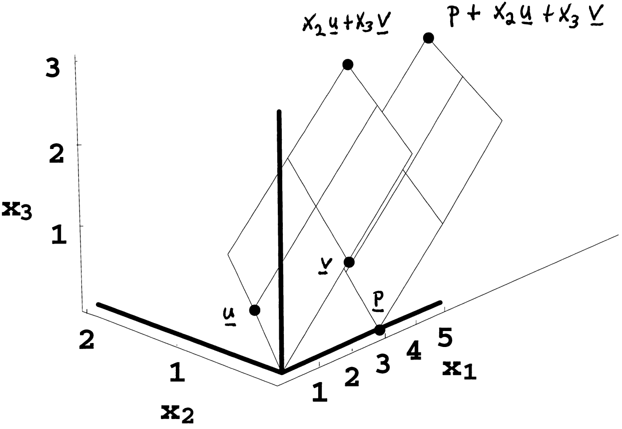
$$\begin{bmatrix} 2 & -4 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 & 3 \end{bmatrix}$$

Vector form of the solution:

$$\overrightarrow{v_2} = \begin{bmatrix} 2x_2 + 2x_3 + 3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\overrightarrow{v_2} = \overrightarrow{v_1} + \overrightarrow{p}$ where $\overrightarrow{p} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Example 7, Graphical Representation



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

Linear Independence, Trivial Solution

A homogeneous system can be viewed a vector equation.

For example, the system

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector equation has the trivial solution ($x_1 = x_2 = x_3 = 0$), but is this the only solution?

Linearly Independence, Definition

Definition A set of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_p}\}$ in $\mathbb{R}^n$ is said to be **linear independence** if the vector equation

$$x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \dots + x_p\overrightarrow{v_p} = \overrightarrow{0}$$

has only the trivial solution.

The set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_p}\}$ is said to be linear dependence if there exists weight $c_1, \dots, c_p$ not all zero, such that

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \dots + c_p\overrightarrow{v_p} = \overrightarrow{0}$$

Linearly Independence, Example

Example 8 Let $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\overrightarrow{v_2} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$ and $\overrightarrow{v_3} = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$

1. Determine if $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly independent.
2. If possible, find a linear dependence relation among $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$.

Solution: (1)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 8, cont.

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable $\rightarrow$ there are nontrivial solutions.

$\therefore \{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly dependent set.

(2) The general solution is

$$\begin{aligned} x_1 &= -33x_3 \\ x_2 &= 18x_3 \\ x_3 &\text{ is free} \end{aligned}$$

Let $x_3 = 1$. Then $x_1 = -33$ and $x_2 = 18$.

Example 8, cont. 2

$$(-33)\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + (18)\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + (1)\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Or

$$(-33)\overrightarrow{v_1} + (18)\overrightarrow{v_2} + (1)\overrightarrow{v_3} = \overrightarrow{0}$$

This is one possible linear dependence relation among $\overrightarrow{v_1}$, $\overrightarrow{v_1}$ and $\overrightarrow{v_1}$ (out of infinitely many linear dependence relation)

Linear Independence of Matrix Columns

A linear dependence relation such as

$$(-33)\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + (18)\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + (1)\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be written as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}\begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\overrightarrow{x} = \overrightarrow{0}$.

Note: The column of matrix A are linearly independence iff the equation $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution.

Special Case

Let's see some special case where the number of vectors is restricted.

A set of one vector

Consider the set containing one vector: $\{\overrightarrow{v_1}\}$.

$\{\overrightarrow{v_1}\}$ is linear independence iff $\overrightarrow{v_1}$ is nonzero.

Because $x_1\overrightarrow{v_1} = \overrightarrow{0}$ has the trivial solution when $\overrightarrow{v_1} = \overrightarrow{0}$. And if $\overrightarrow{v_1} = \overrightarrow{0}$, $\{\overrightarrow{v_1}\}$ is linear dependence since $x_1\overrightarrow{v_1} = \overrightarrow{0}$ has many nontrivial solutions.

A Set of Two Vectors

Example 9 Let

$$\overrightarrow{u_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \overrightarrow{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

1. Determine if $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ is a linearly dependent or a linearly independent set.
2. Determine if $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ is a linearly dependent or a linearly independent set.

Solution: (1) Notice that $\overrightarrow{u_1} = 2\overrightarrow{u_2}$. Therefore

$$2\overrightarrow{u_1} + (-1)\overrightarrow{u_2} = \overrightarrow{0}$$

This means that $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ is linear dependence.

Example 9 cont.

(2) Suppose that

$$c\overrightarrow{v_1} + d\overrightarrow{v_2} = \overrightarrow{0}$$

- Then we have $\overrightarrow{v_1} = (-d/c)\overrightarrow{v_2}$ if $c \neq 0$.
- But this is impossible since $\overrightarrow{v_1}$ is not a multiple of $\overrightarrow{v_2}$
- This means $c = 0$.

Similarly, $\overrightarrow{v_2} = (-c/d)\overrightarrow{v_1}$ if $d \neq 0$

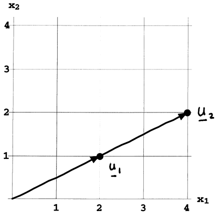
- But this is impossible since $\overrightarrow{v_2}$ is not a multiple of $\overrightarrow{v_1}$, so $d = 0$.

This means that $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ is a linearly independent set.

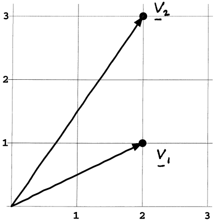
Observation: A set of two vectors is linear dependent iff at least one vector is a multiple of the other.

Observation: A set of two vectors is linearly independent iff neither of the vectors is a multiple of the other.

Example 9, Graphical Representation



linearly dependent



linearly independent

A Set Containing $\overrightarrow{0}$

Theorem A set of vectors $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}\}$ in $\mathbb{R}^n$ containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\overrightarrow{v_1} = \overrightarrow{0}$. Then

$$(1)\overrightarrow{v_1} + (0)\overrightarrow{v_2} + \ldots + (0)\overrightarrow{v_p} = \overrightarrow{0}$$

This show that S is linear dependent.

A Set Containing Too Many Vectors

Theorem: If a set contains more vectors than there are entries in each vector, the the set if linearly dependent. In other words, any set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

- $A\overrightarrow{x} = \overrightarrow{0}$ has more variables than equations.
- $A\overrightarrow{x} = \overrightarrow{0}$ has nontrivial solutions.
- columns of A are linearly dependent.

Sets of Two or More Vectors

Theorem: $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_p}\}$ is linearly dependent iff $\exists\ \overrightarrow{v_j}$, $1 \leq j \leq p$ such that $\overrightarrow{v_j}$ is a linear combination of other vectors.

In fact, if S is linearly dependent and $\overrightarrow{v_1} \neq \overrightarrow{0}$, then some $\overrightarrow{v_j}$, $j > 1$ is a linear combination of the preceding vectors, $\overrightarrow{v_1}, \dots, \overrightarrow{v}_{j-1}$

Example 10: Let $\overrightarrow{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe why a vector $\overrightarrow{w}$ is in $\text{Span}\{\overrightarrow{u}, \overrightarrow{v}\}$ iff $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is linearly dependent.

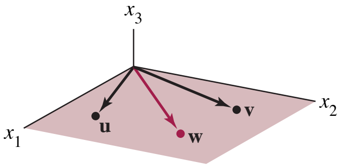
Solution:

- $\overrightarrow{u}$ and $\overrightarrow{v}$ are linearly independent.
  - Since they are not multiple of the other.
  - So they span $\mathbb{R}^3$

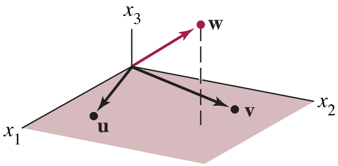
Example 10, cont.

Solution (cont.):

- If $\overrightarrow{w}$ is a linear combination of $\overrightarrow{u}$ and $\overrightarrow{v}$, then by the previous Theorem $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is linear independent.
- Conversely, suppose $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ is linearly dependent. Then by the previous Theorem, there is at least one vector in the set that is a linear combination of the preceding vectors.
- That vector must be $\overrightarrow{w}$ since $\overrightarrow{v}$ is not a multiple of $\overrightarrow{u}$.



Linearly dependent,
w in Span{**u**, **v**}



Linearly independent,
w *not* in Span{**u**, **v**}

Practice Example

Example 11: With the least amount of work possible, decide which of the following sets of vectors are linearly independent or linearly dependence and give a reason for each answer.

Recap

- Solution sets of linear systems
- Homogeneous linear systems
- Nonhomogeneous linear systems
- Parametric vector form
- Linear independence
- Linear dependence
- Special case

Next time we will close the first chapter with linear transformation.