Lecture 14

188 200
Discrete Mathematics and Linear Algebra

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen Uiversity

Overview

Last time we learned how to solve a system of linear equations.

Topics for today.

- Describing a system of linear equation with vectors
- Properties of vectors of linear system
- Linear combination
- A vector equation
- The span of a set of vectors
- A matrix equation

Reference : Section 1.3-1.4

Vector Equations

System of linear equations can be described with the concept and notation of vectors.

Key concepts to master: linear combinations of vectors and a spanning set.

Vector: A matrix with only one column.

Example

Vector in $\mathbb{R}^2$, $\overrightarrow{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Vectors in $\mathbb{R}^n$, (vectors with n entries): $\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Note: We shall denote a vector with $\overrightarrow{u}$.

Algebraic Properties of Vectors

Given two vectors $\overrightarrow{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\overrightarrow{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, their sum is

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

If we have a scalar $c = 2$, $c\overrightarrow{u}$ is

$$2\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

If we have $\overrightarrow{w} = \begin{bmatrix} 0 & 2 \end{bmatrix}$, $u \times w$ is

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 0 & -2 \end{bmatrix}.$$

Note: Multiplication is valid iff the number of columns of the left matrix is the same as the number of rows of the right matrix

Vector Algebraic Rules

For all $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ in $\mathbb{R}^n$ and all scalar c and d:

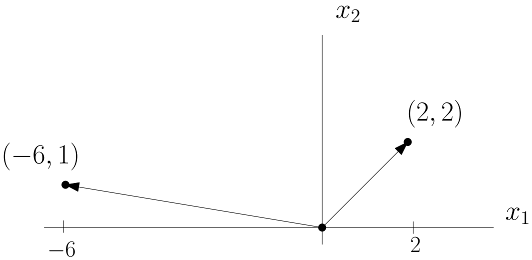
1. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$
2. $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
3. $\overrightarrow{u} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{u} = \overrightarrow{u}$
4. $\overrightarrow{u} + (-\overrightarrow{u}) = -\overrightarrow{u} + \overrightarrow{u} = 0$
5. $c(\overrightarrow{u} + \overrightarrow{v}) = c\overrightarrow{u} + c\overrightarrow{v}$
6. $(c + d)\overrightarrow{u} = c\overrightarrow{u} + d\overrightarrow{u}$
7. $c(d\overrightarrow{u}) = (cd)(\overrightarrow{u})$
8. $1\overrightarrow{u} = \overrightarrow{u}$

Geometric description of Vectors

We can describe vector geometrically.

For example, in $\mathbb{R}^2$, we can described a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ as a point in the plane. And usually drawn as an arrow from the origin (0,0) to the point (a,b).

Example: Let $\overrightarrow{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\overrightarrow{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$. Draw graphs of $\overrightarrow{u}$ and $\overrightarrow{v}$.

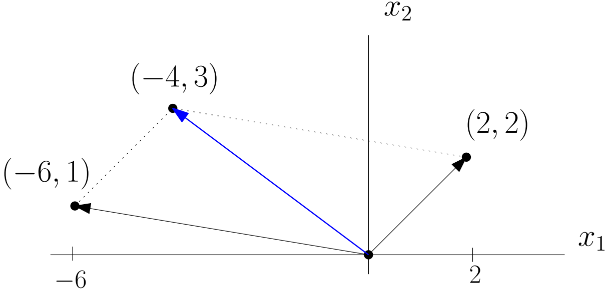


Parallelogram Rules, example

You can also represent an addition of two vectors.

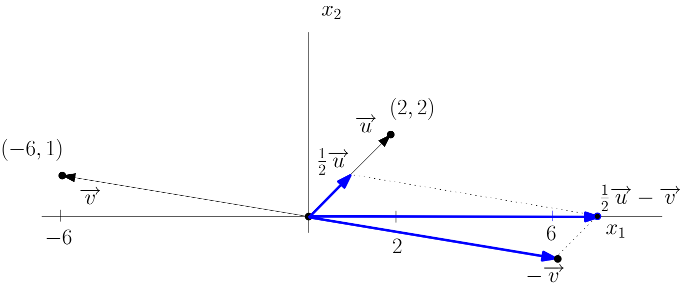
- by using **Parallelogram rule for addition of two vectors:**

Example: Compute $\overrightarrow{u} + \overrightarrow{v}$ then draw graph of the addition.



Parallelogram Rules, example

Example: Compute $\frac{1}{2}\overrightarrow{u} - \overrightarrow{v}$ and then draw the graph of addition.



Linear Combinations

Definition: Given vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$,. . ., $\overrightarrow{v_p}$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$ the vector $\overrightarrow{y}$ defined by

$$\overrightarrow{y} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \ldots + c_p\overrightarrow{v_p}$$

is called **a linear combination** of $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, . . ., $\overrightarrow{v_p}$ with **weights** $c_1, c_2, \ldots, c_p$.

Note: The weights can be any real number including zero.

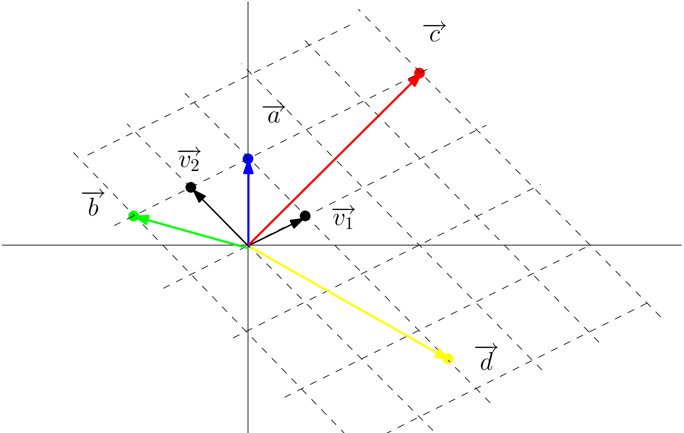
Example: Linear combinations of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$

- $3\overrightarrow{v_1} + 2\overrightarrow{v_2}$
- $(1/3)\overrightarrow{v_1}$
  - $(1/3)\overrightarrow{v_1} + 0\overrightarrow{v_2}$
- $\overrightarrow{v_1} - 2\overrightarrow{v_2}$
- $\overrightarrow{0}$
  - $0\overrightarrow{v_1} + 0\overrightarrow{v_2}$

Example: Let $\overrightarrow{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$:

$\overrightarrow{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \overrightarrow{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \overrightarrow{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \overrightarrow{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

In other words, what are a_1 and b_1 such that $\overrightarrow{a} = a\overrightarrow{v_1} + b\overrightarrow{v_2}$ . . .



Example

Example: Let $\overrightarrow{a_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$,$\overrightarrow{a_2} = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\overrightarrow{a_3} = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$. Determine if $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}$, $\overrightarrow{a_2}$, and $\overrightarrow{a_3}$

Solution: $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ and $\overrightarrow{a_3}$ if we can find weights x_1, x_2, x_3 such that

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + x_3\overrightarrow{a_3} = \overrightarrow{b}$$

Let's write the equation in the form of vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

What linear system does this equation represent?

Linear Combination, Example cont.

Corresponding System:

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & -1 \\ & + & 2x_2 & + & 6x_3 & = & 8 \\ 3x_1 & + & 14x_2 & + & 10x_3 & = & -5 \end{array}$$

Corresponding Augmented Matrix:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{array}{rcr} x_1 & = & 1 \\ x_2 & = & -2 \\ x_3 & = & 2 \end{array}$$

$\overrightarrow{b}$ **is a linear combination of** $\overrightarrow{a_1}$**,** $\overrightarrow{a_2}$ **and** $\overrightarrow{a_3}$ **since**
$\overrightarrow{b} = \overrightarrow{a_1} - 2\overrightarrow{a_2} + 2\overrightarrow{a_3}$.

Linear Combination, Example Review

Review of the last example: $\overrightarrow{a_1}$, $\overrightarrow{a_2}$, $\overrightarrow{a_3}$ and $\overrightarrow{b}$ are columns of the augmented matrix

$$\begin{array}{c} \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \overrightarrow{a_1} & \overrightarrow{a_2} & \overrightarrow{a_3} & \overrightarrow{b} \end{array} \end{array}$$

Solution to

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + x_3\overrightarrow{a_3} = \overrightarrow{b}$$

is found by solving the linear system whose augmented matrix is

$$\begin{bmatrix} \overrightarrow{a_1} & \overrightarrow{a_2} & \overrightarrow{a_3} & \overrightarrow{b} \end{bmatrix}.$$

A Vector Equation

A vector equation

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \ldots + x_n\overrightarrow{a_n} = \overrightarrow{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \overrightarrow{a_1} & \overrightarrow{a_2} & \ldots & \overrightarrow{a_3} & \overrightarrow{b} \end{bmatrix}$$

In particular, $\overrightarrow{b}$ can be generated by a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots \overrightarrow{a_n}$ iff there is a solution to the linear system corresponding to the augmented matrix.

The Span of A Set of Vectors

One of the key idea in Linear Algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}\}$.

Definition:

Suppose $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}$ are in $\mathbb{R}^n$; then **Span**$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}\} =$ set of all linear combinations of

$$\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}.$$

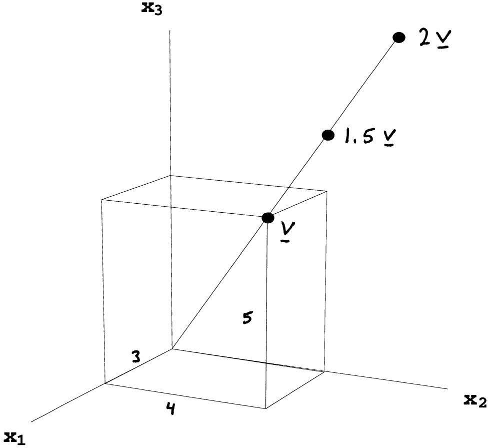
Stated another way: **Span**$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_p}\}$ is the collection of all vectors that can be written as

$$x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \ldots + x_p\overrightarrow{v_p}$$

where $x_1, x_2, \ldots, x_p$ are scalars.

The Span of A Set of Vectors, Example

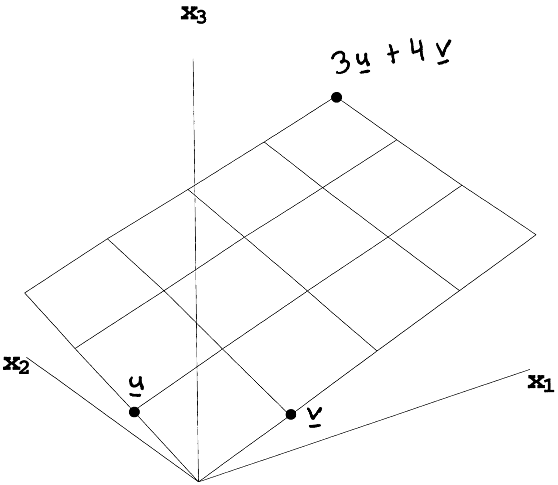
Example: Let $\overrightarrow{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with $\overrightarrow{v}$, $2\overrightarrow{v}$ and $1.5\overrightarrow{v}$ on the graph.



Solution: $\overrightarrow{v}, 2\overrightarrow{v}$ and $1.5\overrightarrow{v}$ all lie on the same line. **Span**$\{\overrightarrow{v}\}$ is the set of all vectors of the form $c\overrightarrow{v}$. Here **Span**$\{\overrightarrow{v}\}$ = a line through the origin.

The Span of A Set of Vectors, Example 2

Example: Label $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{u}+\overrightarrow{v}$ and $3\overrightarrow{u}+4\overrightarrow{v}$ on the graph below.



Solution: $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{u}+\overrightarrow{v}$ and $3\overrightarrow{u}+4\overrightarrow{v}$ all lie in the same plane. **Span**$\{\overrightarrow{u}, \overrightarrow{v}\}$ is the set of all vectors of the form $x_1\overrightarrow{u}+x_2\overrightarrow{v}$. Here, **Span**$\{\overrightarrow{u}, \overrightarrow{v}\}$ = a plane through the origin.

The Span of A Set of Vectors, Example 3

Example: Let $\overrightarrow{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

a. Find a vector in $\textbf{Span}\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$

Solution: For example,

$$2\overrightarrow{v_1} + 3\overrightarrow{v_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

b. Describe $\textbf{Span}\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ geometrically.

Solution: Since $\overrightarrow{v_2}$ is a multiple of $\overrightarrow{v_1}$

$\textbf{Span}\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ is a line through $\overrightarrow{0}$ and $\overrightarrow{v_1}$.

The Span of A Set of Vectors, Example 4

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $\overrightarrow{b}$ in the plane spanned by the columns of A?

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \qquad \overrightarrow{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Do x_1 and x_2 exist so that

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}?$$

The Span of A Set of Vectors, Example 4 cont.

Corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}$$

This shows that the system is inconsistent.

So $\overrightarrow{b}$ is not in the plane spanned by the columns of A.

The Matrix Equation $A\overrightarrow{x} = \overrightarrow{b}$

Linear combinations can be viewed as a matrix-vector multiplication.

Definition If A is an $m \times n$ matrix, with columns $\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}$, and if $\overrightarrow{x}$ is in $\mathbb{R}^n$, then the product of A and $\overrightarrow{x}$, denoted by $A\overrightarrow{x}$, is the linear combination of the columns of A using the corresponding entries in $\overrightarrow{x}$ as weights, i.e.,

$$A\overrightarrow{x} = \begin{bmatrix} \overrightarrow{a_1} & \overrightarrow{a_2} & \ldots & \overrightarrow{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \ldots + x_n\overrightarrow{a_n}$$

The Matrix Equation, Example

Example: Let $\overrightarrow{x} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix}$. Find the product of A and $\overrightarrow{x}$.

Solution:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

The Matrix Equation, Example 2

Example: Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the matrix form $A\overrightarrow{x} = \overrightarrow{b}$ where $\overrightarrow{b}$ is a 3×1 vector.

$$\begin{bmatrix} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{bmatrix}$$

Solution: Corresponding system of equations

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 9 \\ -3x_1 + x_2 &= -2 \end{aligned}$$

Vector equation:

$$x_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

The Matrix Equation, Example 2 cont.

Matrix equation:

$$\begin{array}{ccccc} \begin{bmatrix} 2 & 3 & 4 \\ -3 & 1 & 0 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} & = & \begin{bmatrix} 9 \\ -2 \end{bmatrix} \\ A & \overrightarrow{x} & = & \overrightarrow{b} \end{array}$$

Three equivalent ways of viewing a linear system:

1. as a system of linear equations;
2. as a vector equation $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \ldots + x_n\overrightarrow{a_n} = \overrightarrow{b}$; or
3. as a matrix equation $A\overrightarrow{x} = \overrightarrow{b}$.

Equivalent Ways of Representing A linear System

Theorem 3 If A is a $m \times n$ matrix, with columns $\overrightarrow{a_1}, \ldots, \overrightarrow{a_n}$, and if $\overrightarrow{b}$ is in $\mathbb{R}^m$, then the matrix equation $A\overrightarrow{x} = \overrightarrow{b}$ has the same solution set as the vector equation

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \ldots + x_n\overrightarrow{a_n} = \overrightarrow{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \overrightarrow{a_1} & \overrightarrow{a_2} & \ldots & \overrightarrow{a_n} & \overrightarrow{b} \end{bmatrix}$$

Useful Fact: The equation $A\overrightarrow{x} = \overrightarrow{b}$ has a solution iff $\overrightarrow{b}$ is a linear combination of the columns of A.

Check if $A\overrightarrow{x} = \overrightarrow{b}$ has a solution.

Example: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\overrightarrow{x} = \overrightarrow{b}$ consistent for all $\overrightarrow{b}$?

Solution: Augmented matrix corresponding to $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\overrightarrow{x} = \overrightarrow{b}$ is not consistent for all $\overrightarrow{b}$ since some choices of $\overrightarrow{b}$ make $-2b_1 + b_3$ nonzero.

Check if $A\overrightarrow{x} = \overrightarrow{b}$ has a solution, cont.

$$\begin{array}{cc} A = & \left[\begin{array}{rrr} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{array}\right] \\ & \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \overrightarrow{a_1} & \overrightarrow{a_2} & \overrightarrow{a_3} \end{array} \end{array}$$

The equation $A\overrightarrow{x} = \overrightarrow{b}$ is consistent if

$$-2b_1 + b_3 = 0$$

In other words, $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + x_3\overrightarrow{a_3} = \overrightarrow{b}$ iff $b_3 - 2b_1 = 0$

$A\overrightarrow{x} = \overrightarrow{b}$ has a solution, cont 2

Instead, if any b in $\mathbb{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbb{R}^3$.

Definition We say that the columns of $A = \begin{bmatrix} \overrightarrow{a_1} & \overrightarrow{a_2} & \dots & \overrightarrow{a_p} \end{bmatrix}$ span $\mathbb{R}^m$ if every vector $\overrightarrow{b}$ in $\mathbb{R}^m$ is a linear combination of $\overrightarrow{a_1}, \dots, \overrightarrow{a_p}$ (i.e. $\text{Span}\{\overrightarrow{a_1}, \dots, \overrightarrow{a_p}\}$).

Theorem 4 Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

a. For each $\overrightarrow{b}$ in $\mathbb{R}^m$, the equation $A\overrightarrow{x} = \overrightarrow{b}$ has a solution.

b. Each $\overrightarrow{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Proof of Theorem 4

Proof (outline): Statements (a), (b) and (c) are logically equivalent.

(a) (b) and (c) are clearly equivalent. To complete the proof, we need to show that (a) is true when (d) is true and (a) is false when (d) is false.

Suppose (d) is true. Then row-reduce the augmented matrix $\begin{bmatrix} A & \overrightarrow{b} \end{bmatrix}$:

$$\begin{bmatrix} A & \overrightarrow{b} \end{bmatrix} \sim \ldots \sim \begin{bmatrix} U & \overrightarrow{d} \end{bmatrix}$$

and each row of U has a pivot position and so there is no pivot in the last column of $\begin{bmatrix} U & \overrightarrow{d} \end{bmatrix}$.

So (a) is true.

Now suppose (d) is false. Then the last row of $\begin{bmatrix} U & \overrightarrow{d} \end{bmatrix}$ contains all zeros.

Proof of Theorem 4, cont.

Suppose $\overrightarrow{d}$ is a vector with a 1 as the last entry.

Then $\begin{bmatrix} U & \overrightarrow{d} \end{bmatrix}$ represents an inconsistent system.

Row operations are reversible: $\begin{bmatrix} U & \overrightarrow{d} \end{bmatrix} \sim \ldots \sim \begin{bmatrix} A & \overrightarrow{b} \end{bmatrix}$

$\rightarrow \begin{bmatrix} A & \overrightarrow{b} \end{bmatrix}$ is inconsistent also. So (a) is false.

More Example

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = \overrightarrow{b}$ consistent for all possible $\overrightarrow{b}$?

Solution: A has only 2 columns and therefore has at most 2 pivots.

Since A does not have a pivot in every row, $A\overrightarrow{x} = \overrightarrow{b}$ is not consistent for all possible $\overrightarrow{b}$, according to Theorem 4 d.

Even More Example

Example: Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span R^3?

Solution:

$$\begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- no pivot in row 2
- Again by Theorem 4 d, the column of A, do not span $\mathbb{R}^3$

Properties of Matrix-Vector Product $A\overrightarrow{x}$

Theorem 5

If A is an $m \times n$ matrix, $\overrightarrow{u}$ and $\overrightarrow{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\overrightarrow{u} + \overrightarrow{v}) = A\overrightarrow{u} + A\overrightarrow{v}$

b. $A(c\overrightarrow{u}) = cA\overrightarrow{u}$.

Recap

- Describing a system of linear equation with vectors.
- Properties of vectors
- Linear combination
- A vector equation
- The span of a set of vectors
- A matrix equation

Next time we will have a look at some solution sets of linear system, and linear independence.