Lecture 13

188 200
Discrete Mathematics and Linear Algebra

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Overview

Topics for today.

- A linear algebra: Definition and Applications
- A system of linear equations
- Matrix notation of a system of linear equations
- How do we solve a system of linear equations?
- Existence and uniqueness of a solution

Reference : Section 1.1-1.2

David C. Lay, *Linear Algebra and Its Applications*, third edition.

A Linear Algebra

- What is linear algebra?

  • It is a branch of mathematics concerned with the study of vectors, vector spaces, linear transformations, and systems of linear equations.

- Applications
  - Geometry
  - Image compression
  - Cryptography
  - Economy
  - and many more.

A Linear Equation

A linear equation:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the coefficient $a_1, a_2, \ldots, a_n$ and b are real or complex numbers. And $x_1, x_2, \ldots, x_n$ are unknown variables.

Example:

- $4x_1 - 5x_2 + 2 = x_1$
  - which can be rearranged to $3x_1 - 5x_2 = -2$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$
  - which can be rearranged to $2x_1 + x_2 - x_3 = 2\sqrt{6}$

Not linear:

- $4x_1 - 6x_2 = x_1x_2$
- $x_2 = 2\sqrt{x_1} - 7$

A System of Linear Equations

A system of linear equation (or a **linear system**) is a collection of one or more linear equations involving the same set of variables.

A solution of a linear system is a list of $(x_1, x_2, \ldots, x_n)$ of numbers that makes each equation in the system true.

Example:

1. $x_1 + x_2 = 10$ and $-x_1 + x_2 = 0$
2. $x_1 - 2x_2 = -3$ and $2x_1 - 4x_2 = -6$
3. $x_1 + x_2 = 3$ and $-2x_1 - 2x_2 = -6$

A system of linear equations has either:

1. no solution (inconsistent)
2. exactly one solution (consistent)
3. infinitely many solutions (consistent)

Matrix Notation

A linear system can be recorded as a matrix:

Example:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

can be arranged in matrix which is called a coefficient matrix as follows

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

Or it can be arranged in matrix as an augmented matrix as follows

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Basic: $m \times n$ matrix is a matrix of m rows and n columns.

Each number in a matrix is called entry of matrix and referred to as $a_{i,j}$ where i is row and j is column.

Solving a Linear System

How do we solve a linear system?

Main idea: to replace one system with an equivalent system that is easier to solve.

Note: Two linear system are equivalent iff they have the same solution set.

Example: Solve the following linear system. (Find x_1, x_2 and x_3)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Solving a Linear System: Example

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1)\\ 2x_2 - 8x_3 &= 8 \quad (2)\\ -4x_1 + 5x_2 + 9x_3 &= -9 \quad (3) \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Add 4 times (1) to (3)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1)\\ 2x_2 - 8x_3 &= 8 \quad (2)\\ -3x_2 + 13x_3 &= -9 \quad (3) \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Divide (2) by 2

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \quad (1)\\ x_2 - 4x_3 &= 4 \quad (2)\\ -3x_2 + 13x_3 &= -9 \quad (3) \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Add 2×(2) to (1) and 3×(2) to (3)

Solving a Linear System: Example cont.

$$x_1 - 7x_3 = 8 \qquad (1)$$
$$x_2 - 4x_3 = 4 \qquad (2)$$
$$x_3 = 3 \qquad (3)$$

$$\begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add 7×(3) to (1) and 4×(3) to (2)

$$x_1 = 29 \qquad (1)$$
$$x_2 = 16 \qquad (2)$$
$$x_3 = 3 \qquad (3)$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Note: The example shows how ordinary operations on equations correspond to operations on the augmented matrix.

Question: Can you tell the answer by looking at the last augmented matrix?

Elementary Row Operations

One augmented matrix of a linear system can be transformed to another of an equivalent system using **elementary row operations**.

1. (Replacement) Add one row to a multiple of another rows.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entires in a row by a non-zero constant.

Note: We say two matrix are row equivalent if one can be transformed into the other by elementary row operations.

Note: This is only applicable to rows, not columns!

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Existence and Uniqueness of Solution Set

Given a linear system, the question are

- Does the solution exist? (Is the system consistent?)
- Is the solution unique? (Is the solution the only solution?)

Example: Consider the system of linear

$$x_1 - 2x_2 + 3x_3 = 1 \quad (1)$$
$$-2x_1 + 5x_2 + 4x_3 = -4 \quad (2)$$
$$2x_1 - 8x_3 = 2 \quad (3)$$

the augmented matrix representation of the system is

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ -2 & 5 & 4 & -4 \\ 2 & 0 & -8 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 10 & -2 \\ 0 & 4 & -14 & 0 \end{bmatrix} \quad \rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 10 & -2 \\ 0 & 0 & -54 & 8 \end{bmatrix}$$

Existence and Uniqueness by Echelon Form Matrix

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 10 & -2 \\ 0 & 0 & -54 & 8 \end{bmatrix}$$

Note: Now the matrix is in echelon form which we will formally defined in a minute.

Note: Once the system is in echelon form we can tell if the system is consistent and/or unique. Why?

Solution: The matrix shows that the system is consistent and unique.

- From row 3, we know x_3.
- From row 2, substitute x_3 to get x_2.
- From row 1, substitute x_1 and x_2 to get x_1.
- There is only one x_1, x_2 and x_3.

Inconsistent System of Linear Equations

Example: Is this system consistent?

$$3x_2 - 6x_3 = 8 \quad (1)$$
$$x_1 - 2x_2 + 3x_3 = -1 \quad (2)$$
$$5x_1 - 7x_2 + 9x_3 = 0 \quad (3)$$

Solution: This can be transformed into echelon matrix using elementary row operations as follows

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 3 & 6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The last row shows that $0 \cdot x_3 = -3$. So this system has no solution (inconsistent).

Consistent System, more example

Example: For what real number h will the following system be consistent?

$$3x_1 - 9x_2 = 4 \quad (1)$$
$$-2x_1 + 6x_2 = h \quad (2)$$

Solution: Transform to the following matrix.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 4/3 \\ -2 & 6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 4/3 \\ 0 & 0 & h+8/3 \end{bmatrix}$$

The second equation is $0 \cdot x_1 + 0x_2 = h + 8/3$. Hence the system is consistent only if $h = -8/3$.

Echelon Form

A matrix is on **echelon form** (or **row echelon form**) if

1. All non-zero rows are above any rows of all zeroes.
2. Each leading entry (i.e. left most non-zero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Example:

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

where $\blacksquare$ represent non-zero leading entries and * can be any entires.

Reduced Echelon Form

Matrix is in **reduced echelon form** if it is in echelon form and has the following properties.

1. The leading entry in each non-zero row is 1.
2. Each leading 1 is the only non-zero entry in its column.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Theorem: Each matrix is row-equivalent to one and only one reduced echelon matrix.

Important Terms

- **Pivot position:** a position of a leading entry in an echelon form of the matrix.
- **Pivot:** a non-zero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.
- **Pivot column:** a column that contains a pivot position.

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 3 & 6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Row Reduced, Example

Example: Row reduced the following matrix to echelon form and locate the pivots and the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} \mathbf{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -2 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & \mathbf{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Row Reduced, Example cont.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 4 & 5 & -9 & -7 \\ 0 & \mathbf{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \mathbf{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

- 1, 2 and -5 are pivot.
- Columns 1,2 and 4 are pivot columns.

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

Note: Henceforth, we shall denote two matrices that are row equivalent matrix with $\sim$.

Reduced Echelon Form, Example

Example: Row reduce the following matrix to reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

At this point we have matrix in echelon form.

Reduced Echelon Form, Example cont.

Final step to create the reduced echelon form:

Beginning with the rightmost or leftmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot column are 1,2 and 5.

Solution of Linear Systems

- **Basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **Free variable:** all nonbasic variables.

Example:

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 + 6x_2 \quad + 3x_4 \quad &= 0 \\ x_3 - 8x_4 \quad &= 5 \\ x_5 &= 7 \end{aligned}$$

pivot columns: 1,3,5

basic variables: x_1, x_3, x_5

free variables: x_2, x_4

Solving a Consistent Linear System

Final step in solving a consistent in linear system: After the augmented matrix is in reduced echelon form and the system is written down as a set of equations:

solve each equation for the basic variable variable in terms of the free variables (if any) in the equation.

Solution

$$\begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 8x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

$$\begin{cases} x_2 \text{ is free.} \\ x_4 \text{ is free.} \\ x_1 = -6x_2 - 3x_4 \\ x_3 = 5 + 8x_4 \\ x_5 = 7 \end{cases}$$

Solving a Linear System With Free Variables.

The **general solution** of the system provide a parametric description of the solution set. (The free variable acts as parameters.)

The above system has **infinitely many solutions**.

Why?

x_2 and x_4 are free variables.

- which implies that x_2 and x_4 are free to be any values.

Warning: Use only the reduced echelon form to solve a system.

Existence and Uniqueness Questions

Example: Consider the following linear system.

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= 0 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

The augmented matrix of the system can be transformed into the echelon form as

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Notice: The right most column is not a pivot column, hence the system is consistent.

Existence and Uniqueness Questions, Example cont.

The reduced echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Basic variables: x_1, x_2 and x_5

Free variables: x_3 and x_4

Hence we have,

- $x_5 = 4$
- $x_1 - 2x_3 + 3x_4 = -24$
  - $\rightarrow x_1 = -24 + 2x_3 - 3x_4$
- $x_2 - 2x_3 + 2x_4 = -7$
  - $\rightarrow x_2 = -7 + 2x_3 - 2x_4$

Example

Example: Consider the following linear system:

$$\begin{aligned} 3x_1 + 4x_2 &= -3 \\ 2x_1 + 5x_2 &= 5 \\ -2x_1 - 3x_2 &= 1 \end{aligned}$$

$$\begin{bmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 3x_1 + 4x_2 &= -3 \\ x_2 &= 3 \end{aligned}$$

Solution: The echelon form matrix show that the system is consistent. And since there is no free solution, we have a unique solution.

- $x_2 = 3$, $x_1 = -5$

Existence and Uniqueness

We can conclude the existence and uniqueness of a linear system by the following theorem.

Theorem: Existence and Uniqueness

1. A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column, i.e., iff an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ where } b \text{ is non-zero.}$$

2. If a linear system is consistent, then the solution contains either
    2.1 a unique solution (when there are no free variables) or
    2.2 infinitely many solutions (when there is at least one free variable).

Using Row Reduction to Solve Linear System.

We can conclude the algorithm of solving a linear system using row reduction as follows:

Solving linear system by row reduction:

1. Write the augmented matrix of the system.
2. Row reduce the matrix to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.

Example

Example:

a. What is the largest possible number of pivots 4×6 matrix can have? Why?

4. There is at most one pivot per row.

b. What is the largest possible number of pivots a 6×4 matrix can have? Why?

4. There is at most one pivot per column.

c. How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

Since the corresponding augmented matrix has 3 rows, there are at most 3 pivots. But since there are 4 unknowns, a free variable exists, so there are infinitely many solutions.

Example cont.

Example: cont.

d. Suppose the coefficient matrix corresponding to a linear system is 4×6 and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?

Since the system is inconsistent, there is a pivot in the last column. So 4 pivot columns exist.

Recap

- Introduction to a linear algebra: Definition and Applications
- A system of linear equations
- Matrix notation of a system of linear equations
- How to solve a system of linear equations
- Existence and uniqueness of a solution
- Echelon form matrix and reduced echelon form matrix
- Next lecture we will discuss vectors and vector spaces