# Lecture 12

## 188 200
## Discrete Mathematics and Linear Algebra

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# Overview

**Topics for today.**

- Definition of relations
- A relation on a single set.
- Properties of relation
  - Reflexive
  - Symmetric
  - Transitive
- n-ary relations
- Equivalence relation
- Equivalence class
- Recap

**Reference :** Section 10.1-10.3

# Definition of Relations

Binary relations establish a relationship between elements of two sets.

**Definition:** Let $A$ and $B$ be non-empty sets. A binary relation from $A$ to $B$ is a subset of $A \times B$.

In other words, a binary relation $R$ is a set of ordered pairs $(a_i, b_i)$ where $a_i \in A$ and $b_i \in B$.

**Notation:** we say that

- $a \; R \; b$ if $(a, b) \in R$
  - $a \; R \; b$ is read "$a$ is realted to $b$ by $R$".
  - I will be using both $a \; R \; b$ and $(a, b) \in R$.
- $a \not R \; b$ if $(a, b) \notin R$

# Relations, Example

**Example:** Assume that Mike and Pete are taking 188 200. Mike is also taking 188 230. Moreover, Jane is taking 188 210 and 000 103. Define a relation $R$ that represents the relationship between people and classes.

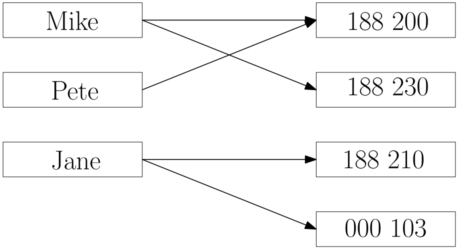
**Solution:**

- Let $P$ denote a set of people, so $P = \{\text{Mike, Pete, Jane}\}$.
- Let $C$ denote a set of classes, so $C = \{188\ 200, 188\ 230, 188\ 210, 000\ 103\}$
- By definition $R \subseteq P \times C$.
- From the above statement, we know that
  - $(\text{Mike}, 188\ 200) \in R$
  - $(\text{Pete}, 188\ 200) \in R$
  - $(\text{Mike}, 188\ 230) \in R$
  - $(\text{Jane}, 188\ 210) \in R$
  - $(\text{Jane}, 000\ 103) \in R$
- So, $R = \{\ (\text{Mike}, 188\ 200), (\text{Pete}, 188\ 200), (\text{Mike}, 188\ 230), (\text{Jane}, 188\ 210), (\text{Jane}, 000\ 103)\ \}$

# Relations: a diagram

A relation can be represented with a diagram.

Recall from the previous example that
$R = \{$ (Mike, 188 200), (Pete, 188 200), (Mike, 188 230), (Jane, 188 210), (Jane, 000 103) $\}$.



# Relations: a table

A relation can also be represented as a table.

Recall from the previous example that
$R = \{$ (Mike, 188 200), (Pete, 188 200), (Mike, 188 230), (Jane, 188 210), (Jane, 000 103) $\}$.

| R | 188 200 | 188 230 | 188 210 | 000 103 |
|---|---|---|---|---|
| Mike | X | X | | |
| Pete | X | | | |
| Jane | | | X | X |

# Function or Relation?

So does this mean that relations are the same as functions?

**Definition:** Let $A$ and $B$ be non-empty sets. A function $f$ from $A$ to $B$ is a relation that satisfies the following properties

1. $\forall x \in A$, $\exists y \in B | (x, y) \in f$
2. $\forall x \in A$, and $\forall y, z \in B$, if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$.

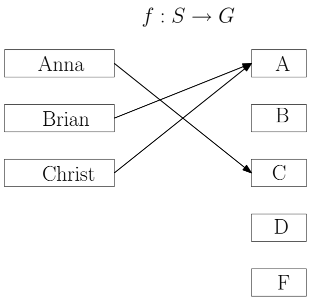
- This would mean that for example a person can enroll in only one class.

Reconciling this with our definition of a relation, we see that

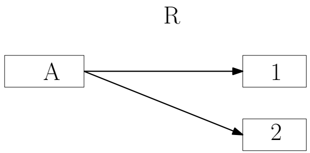
- Every function is also a relation
- Not every relation is a function

# Relations: Example 2.

- Consider $f : S \rightarrow G$
  - Clearly a function.
  - Can also be represented as the relation, $R =$ {(Anna, C), (Brian, A), (Christ, A)}




- Consider the set $R = \{(A, 1), (A, 2)\}$
  - Clearly a relation
  - Cannot be represented as a function!



# Relations: on a single set

We can define binary relations on a single set.

**Definition:** A binary relation on a set $A$ is a binary relation from $A$ to $A$. That is, a relation on the set $A$ is a subset of $A \times A$.

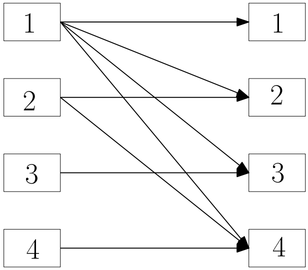
**Example:** Let $A$ be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$

- 1 divides everything.
- 2 divides itself and 4.
- 3 divides itself.
- 4 divides itself.

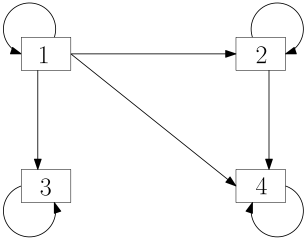
- So, $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

# Representing the last example as a graph

Recall: $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$




If we are talking about a relation on a set $A$, it is not necessary to draw set $A$ twice.



# Now let's test what you have learned.

**Question:** Which of the following relations contain each of the pairs $(1,1)$, $(1,2)$, $(2,1)$, $(1,-1)$, and $(2,2)$?

- $R_1 = \{(a,b) | a \leq b\}$
- $R_2 = \{(a,b) | a > b\}$
- $R_3 = \{(a,b) | a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) | a = b\}$
- $R_5 = \{(a,b) | a = b + 1\}$
- $R_6 = \{(a,b) | a + b \leq 3\}$
- These are all relations on an infinite set!!

| | (1,1) | (1,2) | (2,1) | (1,-1) | (2,2) |
|---|---|---|---|---|---|
| $R_1$ | | | | | |
| $R_2$ | | | | | |
| $R_3$ | | | | | |
| $R_4$ | | | | | |
| $R_5$ | | | | | |
| $R_6$ | | | | | |

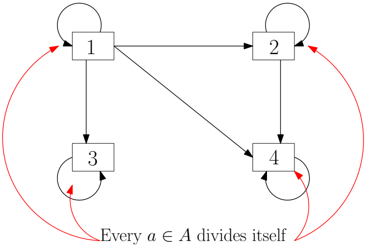
# Properties of Relations: Reflexive

**Definition:** A relation $R$ on a set $A$ is reflexive if $(a, a) \in R$ for every $a \in A$.

Let's look back at our "divide" example.

**Example:** Let $A$ be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$

Recall that
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$.




| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | X | X | X | X |
| **2** | | X | | X |
| **3** | | | X | |
| **4** | | | | X |

# Properties of Relations: Symmetric

**Definition:** A relation $R$ on a set $A$ is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for every $a, b \in A$. If $R$ is a relation in which $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$, we say that $R$ is antisymmetric.

**Mathematically:**

- Symmetric: $\forall a \forall b | (a, b) \in R \rightarrow (b, a) \in R)$.
- Antisymmetric: $\forall a \forall b | ((a, b) \in R \land (b, a) \in R) \rightarrow (a = b)$.
    - Equivalently $\forall a \forall b | ((a, b) \in R \land a \neq b \rightarrow (b \not R\ a)$

**Examples:**

- Symmetric: $R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)\}$
- Antisymmetric: $R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)\}$

# Symmetric and Antisymmetric Relations

R = {(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (4,4)}

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | X | X | X | X |
| 2 | X | | X | |
| 3 | X | X | | |
| 4 | X | | | X |

## Symmetric relation

- Diagonal axis of symmetry
- Not all elements on the axis of symmetry need to be included in the relation.

R = {(1,1), (1,2), (1,3), (1,4), (2,4), (3,3), (4,4)}

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | X | X | X | X |
| 2 | | | | X |
| 3 | | | X | |
| 4 | | | | X |

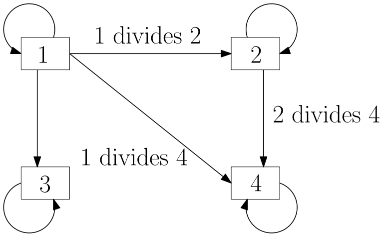
## Asymmetric relation

- No axis of symmetry
- Only symmetry occurs on diagonal
- Not all elements on the diagonal need to be included in the relation

# Properties of Relations: Transitive

**Definition:** A relation $R$ on a set $A$ is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for every $a, b, c \in A$.

Let's look back at our "divide" example. Recall that that $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$ is transitive.




This isn't terribly interesting, but it is transitive nonetheless.

More common transitive relations include equality and comparison operators like $<$, $>$, $\leq$, and $\geq$.

# Properties of Relations, Example

**Example:** Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

- $R_1 = \{(a, b) | ab\}$
- $R_2 = \{(a, b) | a > b\}$
- $R_3 = \{(a, b) | a = b \text{ or } a = -b\}$
- $R_4 = \{(a, b) | a = b\}$
- $R_5 = \{(a, b) | a = b + 1\}$
- $R_6 = \{(a, b) | a + b3\}$

| | Reflexive | Symmetric | Antisymmetric | Transitive |
|---|---|---|---|---|
| $R_1$ | | | | |
| $R_2$ | | | | |
| $R_3$ | | | | |
| $R_4$ | | | | |
| $R_5$ | | | | |
| $R_6$ | | | | |

# Relations can be combined using set operations

**Example:** Let $R$ be the relation that pairs students with courses that they have taken. Let $S$ be the relation that pairs students with courses that they need to graduate. What do the relations $R \cup S$, $R \cap S$, and $S - R$ represent?

**Solution:**

- $R \cup S =$ All pairs $(a, b)$ where
  - student $a$ has taken course $b$ OR
  - student $a$ needs to take course $b$ to graduate
- $R \cap S =$ All pairs $(a, b)$ where
  - student $a$ has taken course $b$ AND
  - student $a$ needs course $b$ to graduate
- $S - R =$ All pairs $(a, b)$ where
  - student $a$ needs to take course $b$ to graduate BUT
  - student $a$ has not yet taken course $b$

# Relations can be combined using functional composition

**Definition:** Let $R$ be a relation from the set $A$ to the set $B$, and $S$ be a relation from the set $B$ to the set $C$. The composite of $R$ and $S$ is the relation of ordered pairs $(a, c)$, where $a \in A$ and $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $R \ o \ S$.

**Example:** What is the composite relation of $R$ and $S$?

$R : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$

$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$

$S : \{1, 2, 3, 4\} \rightarrow \{0, 1, 2\}$

$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$

So: $R \ o \ S = \{(1, 0), (3, 0), (1, 1), (3, 1), (2, 1), (2, 2)\}$

# We can also "relate" elements of more than two sets

**Definition:** Let $A_1$, $A_2$, ..., $A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1$, $A_2$, ..., $A_n$ are called the domains of the relation, and $n$ is its degree.

**Example:** Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ consisting of triples $(a, b, m)$ where $a \equiv b (\text{mod } m)$; $m|(a - b)$.

- What is the degree of this relation?
- What are the domains of this relation?
- Are the following tuples in this relation?
  - (8,2,3)
  - (-1,9,5)
  - (11,0,6)

# Equivalence Relation

Informally: An equivalence relation partitions elements of a set into classes of "equivalent" objects.

Formally: A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

Reflextive: How can a relation define equivalent objects if an element isn't equivalent to itself?

Transitive: If $x$ is equivalent to $y$, and $y$ is equivalent to $z$, shouldn't $x$ also be equivalent to $z$?

Symmetric: If $x$ is equivalent to $y$, shouldn't $y$ be equivalent to $x$?

**Definition:** Two elements $a$ and $b$ that are related by some equivalence relation are called equivalent. We denote this by $a \sim b$ (or $a \sim_R b$).

# Relations, Example

**Example:** Let $R$ be the relation on the set of integers such that $a \; R \; b$ iff $a = b$. Is $R$ an equivalence relation?

Intuition says yes, so lets verify:

- Is $R$ symmetric?
- Is $R$ reflexive?
- Is $R$ transitive?

Conclusion: Since $R$ is symmetric, reflexive, and transitive, we know that $R$ is an equivalence relation.

# Congruence Modulo $m$

**Example:** Let $m$ be a positive integer greater than 1. Show that $R = \{(a, b) | a \equiv b (\text{mod } m)\}$ is an equivalence relation.

**Solution:**

- Recall: $a \equiv b(\text{mod } m) \leftrightarrow m|(a - b)$
- Is $R$ reflexive?
  - $a \equiv a(\text{mod } m) \leftrightarrow m|(a - a)$.
  - Yes, $R$ is reflexive since $m|0$.
- Is $R$ symmetric?
  - If $a \equiv b(\text{mod } m)$, then $m|(a - b)$, so $(a - b) = km$ for some integers $k$.
  - Note that $(b - a) = -km$. So $m|(b - 1)$.
  - So $b \equiv a(\text{mod } m)$ and $R$ is symmetric.
- Is $R$ transitive?
  - $a \equiv b(\text{mod } m) \leftrightarrow (a - b) = km$, so $a = km + b$.
  - $b \equiv c(\text{mod } m) \leftrightarrow (b - c) = rm$, so $c = brm$.
  - Note that $ac = (km + b)(brm) = km + rm = (k + r)m$.
  - Since $m|(a - c), a \equiv c(\text{ mod } m)$, and $R$ is transitive.
- Conclusion: $R$ is an equivalence relation.

# What about the divides relation?

**Example:** Is the “divides” relation on positive integers an equivalence relation?

**Solution:**

- Reflexive?
- Symmetric?
- Transitive?

- Conclusion: Since the “divides” relation is not symmetric, it cannot be an equivalence relation.

# String Length

**Example:** Suppose that $R$ is the relation on the set of strings of English letters such that $a\ R\ b$ iff $\ell(a) = \ell(b)$, where $\ell(x)$ is the length of string $x$. Is $R$ an equivalence relation?

**Solution:**

- Reflexive?
- Symmetric?
- Transitive?

- Conclusion: $R$ is an equivalence relation.

# Magnitude of differences

**Example:** Let $R$ be the relation on the set of real numbers such that $x\,R\,y$ iff $x$ and $y$ are real numbers that differ by less than 1, i.e., $|x - y| < 1$. Is $R$ an equivalence relation?

**Solution:**

- First, a few test cases:
  - 1.1 $R$ 2.0?
  - 1.1 $R$ 3.0?
  - 2.0 $R$ 2.5?
- Reflexive?
- Symmetric?
- Transitive?

- Conclusion: Since $R$ is not transitive, it cannot be an equivalence relation.

# Equivalence Class

**Definition:** Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to some element $a$ is called the equivalence class of $a$.

**Note:** We denote the equivalence class of element $a$ under relation $R$ as $[a]_R$. If only one relation is being considered, we can drop the subscript and denote the equivalence class of $a$ as $[a]$.

**Example:** What are the equivalence classes of 0 and 1 under congruence modulo 4?

- $m$ and $n$ and $m$ are called congruent modulo $k$ if $m \mod k = n \mod k$.
- $[0]$ contains all integers $x$ such that $x \equiv 0 \pmod 4$
- $[1]$ contains all integers $x$ such that $x \equiv 1 \pmod 4$
- So $[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$
- And $[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

# Variable names in C

**Example:** Some compilers for the C programming language truncate variable names after the first 31 characters. As a result, any two variable names that agree in the first 31 characters are considered to be identical. What are the equivalence classes of the variable names "Number_of_tropical_storms", "Number_of_named_tropical_storms", and "Number_of_named_tropical_storms_in_the_Atlantic_in_2005"?

**Solution:**

- "Number_of_tropical_storms" =
- "Number_of_named_tropical_storms" =
- "Number_of_named_tropical_storms_in_the_Atlantic_in_2005" =

# An equivalence relation divides a set into disjoint subsets

**Example:** At KKU, a student can either major in computer science or art history, but not both. Let $R$ be the relation defined such that $a \ R \ b$ if $a$ and $b$ are in the same major.

**Observations:**

- $R$ is an equivalence relation (Why?)
- $R$ breaks the set $S$ of all students into two subsets:
  - $C =$ Students majoring in computer science.
  - $A =$ Students majoring in art history.
- No student in $C$ is also in $A$.
- No student in $A$ is also in $C$.
- $C$ and $A$ are equivalence classes of $S$.

# Equivalence classes are either equal or disjoint

**Theorem:** If $R$ is an equivalence relation on some set $A$, then the following three statements are equivalent:

1. $a\ R\ b$
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$.

**Proof:**

- To prove this, we'll prove that $(1) \rightarrow (2)$, $(2) \rightarrow (3)$, and $(3) \rightarrow (1)$.
- $(1) \rightarrow (2)$
  - Assume that $a\ R\ b$
  - To prove that $[a] = [b]$, we will show that $[a] \subseteq [b]$ and $[b] \subseteq [a]$.
  - Suppose that $c \in [a]$, then $a\ R\ c$.
  - Since $aRb$ and $R$ is symmetric, we have that$b\ R\ a$.
  - Since $R$ is transitive, we have that $b\ R\ a$ and $a\ R\ c$, so $b\ R\ c$.
  - This means that $c \in [b]$ and thus that $[a] \subseteq [b]$.
  - The proof that $[b] \subseteq [a]$ is identical.

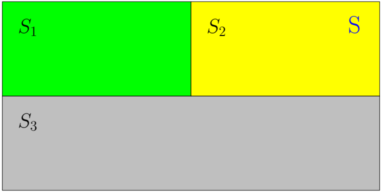
# Equivalence classes are either equal or disjoint, cont.

**Proof (cont.):**

- $(2) \rightarrow (3)$
  - Assume that $[a] = [b]$
  - $[a] \cap [b]$ is non-empty since $a \in [a]$
- $(3) \rightarrow (1)$
  - Assume that $[a] \cap [b] \neq \emptyset$
  - This means that there exists some element $c \in [a] \cap [b]$
  - So, $a \ R \ c$ and $b \ R \ c$.
  - By symmetry, we have that $c \ R \ b$.
  - By transitivity, we have that $a \ R \ c$ and $c \ R \ b$ means $a \ R \ b$.
- Since $(1) \rightarrow (2)$, $(2) \rightarrow (3)$, and $(3) \rightarrow (1)$, all three statements are equivalent.

# Equivalence classes partition a set

**Definition:** A partition of a set $S$ is a collection of disjoint subsets that have $S$ as their union.




**Observation:** The equivalence classes of a set partition that set.

- $\cup_{a \in A}[a] = S$ since each $a \in A$ is in its own equivalence class
- By our theorem, we know that either $[a] = [b]$, or $[a] \cap [b] = \emptyset$.

# The integers (mod m), redux

**Example:** What are the sets in the partition produced by the equivalence relation equivalence mod 4?

**Solution:**

- $[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$
- $[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$
- $[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$
- $[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

- Note that each integer is in one of these sets, and each set is disjoint. Thus, these equivalence classes partition the set Z.

# Conversely, a partition of a set describes an equivalence relation

**Example:** List the ordered pairs in the equivalence relation $R$ produced by the partition $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

**Solution:**

- From $A = \{1, 2, 3\}$ we have
  - $(1, 1), (1, 2), (1, 3) \in R$
  - $(2, 1), (2, 2), (2, 3) \in R$
  - $(3, 1), (3, 2), (3, 3) \in R$
- From $B = \{4, 5\}$ we have
  - $(4, 4), (4, 5) \in R$
  - $(5, 4), (5, 5) \in R$
- From $C = \{6\}$ we have
  - $(6, 6) \in R$

# Recap

- Definition of relations
- A relation on a single set.
- Properties of relation
  - Reflexive
  - Symmetric
  - Transitive
- n-ary relations
- Equivalence relation
- Equivalence class