Lecture 11

188 200
Discrete Mathematics and Linear Algebra

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Overview

Topics for today.

- Functions
- One-to-one functions
- Onto functions
- Inverse functions
- Composition of functions
- The pigeonhole principle

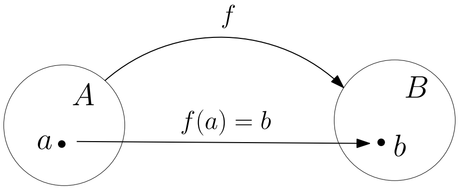
Reference : Section 7.1-7.4

Definition of Functions

Definition: Let A and B be non-empty sets. A function f from A to B is an assignment of exactly one element of set B to each element of set A.

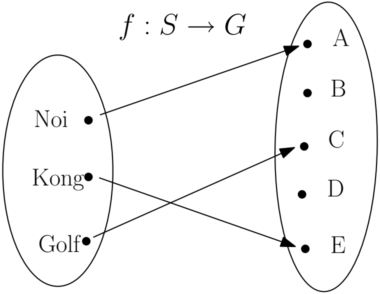
Note: We write $f : A \rightarrow B$ to denote that f is a function from A to B.

Note: We say that $f(a) = b$ if the element $a \in A$ is mapped to the unique element $b \in B$ by the function f.



Functions can be defined in many ways

1. Explicitly
   - $f : \mathbb{Z} \to \mathbb{Z}$
   - $f(x) = x^2 + 2x + 1$
2. Using a programing language
   - int min(int x, int y) $=$ {if $x < y$ return x: return y}
3. Using relation
   - Let $S = \{Noi, Kong, Golf\}$
   - Let $G = \{A, B, C, D, E\}$



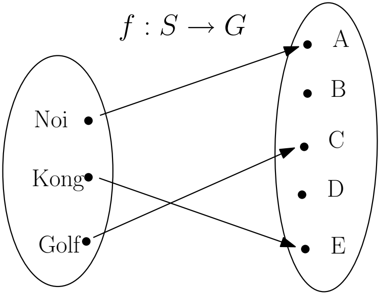
More terminology

$f : S \to G$

The domain of a function is the set that the function maps from, while the codomain is the set that is mapped to.

If $f(a) = b$, b is called the image of a, and a is called the preimage of b.

The range of a function $f : S \to G$ is the set of all images of elements of S.



Domain, Codomain and Range.

What are the domain, codomain and range of the following functions?

1. $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x^3$
   - Domain:
   - Codomain:
   - Range:
2. $g : R \rightarrow R, g(x) = x - 2$
   - Domain:
   - Codomain:
   - Range:
3. int foo(int x, int y) = { return (x*y)%2; }

   % is mod as you might have seen this in Java or C
   - Domain:
   - Codomain:
   - Range:

Function or non-function?

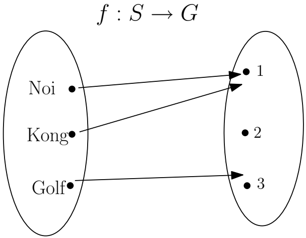
Which of the following diagram define function from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?

One-to-one Functions.

Definition: A function $f : A \to B$ is one-to-one, or injective, iff $\forall x, y \in A[(f(x) = f(y)) \to (x = y)]$.

Are the following functions injections?

- $f : R \to R, f(x) = x + 1$
- $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$
- $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \sqrt{x}$
- $f : S \to G$



Conclustion: A one-to-one function never assigns the same image to two different elements.

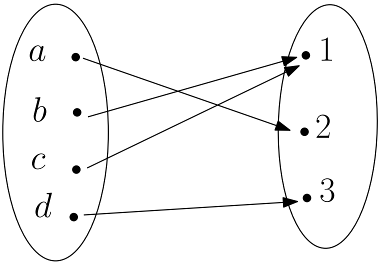
Onto Functions

Definition: We call a function $f : A \rightarrow B$ onto, or surjective, iff for every element $b \in B$, there is some element $a \in A$ such that $f(a) = b$.

Note: Think about an onto function as "covering" its codomain entirely. In other words, the a function is onto if codomain is equal to range.

The following function is a surjection (onto function):

$$f : A \rightarrow B$$

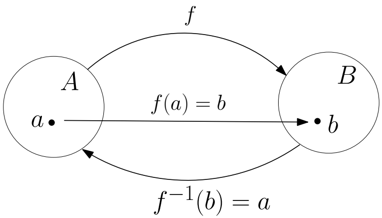


Are the following functions one-to-one, onto, both, or neither

Inverse Functions

Definition: If $f : A \to B$ is a bijection, the inverse of f is the function $f^{-1} : B \to A$ that assigns to each $b \in B$ the unique value $a \in A$ such that $f(a) = b$. That is, $f^{-1}(b) = a$ iff $f(a) = b$.

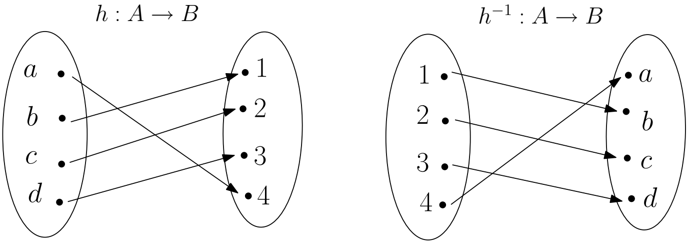
Graphically:



Note: Only a bijection can have an inverse. (Why?)

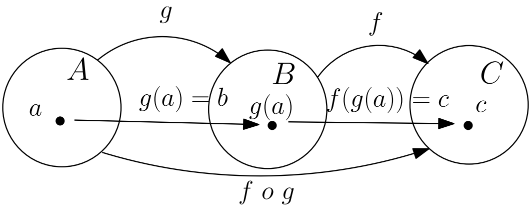
Do the following functions have inverses?

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$
2. $g : \mathbb{Z} \to \mathbb{Z}, g(x) = x + 1$
3. $h : A \to B$



Composition of Functions

Definition: Given two functions $g : A \to B$ and $f : B \to C$, the composition of f and g, denoted $f\ o\ g$, is defined as $(f\ o\ g)(x) = f(g(x))$.



Note: For $f\ o\ g$ to exist, the domain of f must be a subset of the codomain of g.

Can the following functions be composed? If so, what is their composition?

Let $f : A \rightarrow A$ such that $f(a) = b$, $f(b) = c$, $f(c) = a$
$g : B \rightarrow A$ such that $g(1) = b$, $g(4) = a$

1. $(f \ o \ g)(x)$?
2. $(g \ o \ f)(x)$?

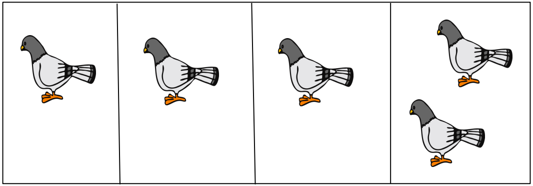
Let $f : Z \rightarrow Z, f(x) = 2x + 1$
$g : Z \rightarrow Z, g(x) = x2$

1. (f o g)(x)?
2. (g o f)(x)?

Note: There is never a guarantee that $(f \ o \ g)(x) = (g \ o \ f)(x)$.

The Pigeonhole Principle

The pigeonhole principle: If k is a positive integer and $k + 1$ objects are placed in k boxes, then at least one box contains at least two objects.



Note: The pigeonhole principle is an incredibly simple concept that is extremely useful!

The pigeonhole principle is also easy to prove

The pigeonhole principle: If k is a positive integer and $k+1$ objects are placed in k boxes, then at least one box contains at least two objects.

Proof: Assume that each of the k boxes contains at most 1 item. This means that there are at most k items, which is a contradiction of our assumption that we have $k+1$ items, so at least one box must contain more than one item.

Q.E.D.

The pigeonhole principle: Examples

Example: Among any group of 367 people there are at least two with the same birthday, since there are only 366 possible birthdays.

Example: Among any 27 English words, at least two will start with the same letter.

Using the pigeonhole principle to prove an interesting results

Example: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If we choose five integers from A, must at least one pair of the integers have sum of 9?

Example:

More General Form of the Pigeonhole Principle.

There is a more general form of the pigeonhole principle that is even more useful

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ items.

Proof:

- Assume that no box contains more than $\lceil N/k \rceil - 1$ objects.
- Note that $\lceil N/k \rceil < (N/k) + 1$.
- So, $k(\lceil N/k \rceil - 1) < k((N/k + 1) - 1) = N$.
- This contradicts our assumption that we had N objects.

Q.E.D.

The Pigeonhole Principle, Example

Example: What is the minimum number of students needed to at least have six students receive the same grade, if possible grades are A, B, C, D, and F?

Solution:

- Need the smallest integer N such that $\lceil N/5 \rceil = 6$.
- With 25 students, it would be possible (though unlikely) to have 5 students get each possible grade.
- By adding a 26th student, we guarantee that at least 6 students get one possible grade.
- So, the smallest such N is $5(5) + 1 = 26$.

Q.E.D.

A Card Game Using The Pigeonhole Principle

Example: How many cards must be drawn from a standard 52-card deck to guarantee that three cards of the same suit are drawn?

Solution:

- Lets make 4 piles: one for each suit.
- Let N be the number of cards that must be drawn to guarantee three cards of the same suit are achieved.
- We want to have $\lceil N/4 \rceil \geq 3$.
- We can do this using $4(2) + 1 = 9$ cards

Q.E.D.

Recap

- Definition of functions
- One-to-one functions
- Onto-functions
- Inverse functions
- Composition of functions
- The pigeonhole principle

Next time, we will discuss a closely related topic to function: Relation!