Lecture 8

178 359
Simulation and Modeling

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Overview

- Relationship between Bezier and Said-Ball points
- Relationship between Bezier and Wang-Ball points
- Computation aspect

Surface Model

- Representing surfaces
- Tensor product
- A plane
- Finding a point on a surface

Relationship Bezier and Said-Ball Points

The Bezier control points b_k of a Said-Ball curve of degree n can be expressed in terms of Said-Ball control points v_i as follows:

$$b_k = \begin{cases} \sum_{i=0}^{k} \frac{\binom{\lfloor n/2 \rfloor + i}{\lfloor n/2 \rfloor}\binom{\lfloor (n-1)/2 \rfloor - i}{k-i} v_i}{\binom{n}{k}}, & i < n/2 \\ \\ \sum_{i=0}^{n+k} \frac{\binom{\lfloor n/2 \rfloor + i}{\lfloor n/2 \rfloor}\binom{\lfloor (n-1)/2 \rfloor - i}{n-k-i} v_{n-i}}{\binom{n}{k}}, & i > n/2 \end{cases}$$

$b_{n/2} = v_{n/2}$. for n is even.

Note: This is feasible for both odd and even degree.

Relationship Bezier and Said-Ball Points 2

The Said-Ball control points v_k of a Bezier curve of degree n can be expressed in terms of Bezier control points b_i as follows:

$$v_0 = b_0$$
$$v_n = b_n$$

For $i = 1$ to $\lfloor (n-1)/2 \rfloor$

$$v_i = \left[\binom{n}{i} b_i - \sum_{j=0}^{i-1} \binom{\lfloor n/2 \rfloor + j}{\lfloor n/2 \rfloor} \binom{\lfloor (n-1)/2) \rfloor - j}{i-j} v_j / \binom{\lfloor n/2 + i \rfloor}{\lfloor n/2 \rfloor} \right]$$
$$v_{n-i} = \left[\binom{n}{i} b_{n-i} - \sum_{j=0}^{i-1} \binom{\lfloor n/2 \rfloor + j}{\lfloor n/2 \rfloor} \binom{\lfloor (n-1)/2) \rfloor - j}{i-j} v_{n-j} / \binom{\lfloor n/2 + i \rfloor}{\lfloor n/2 \rfloor} \right]$$

If n is even then $v_{n/2} = b_{n/2}$.

Relationship Bezier and Said-Ball Points Table

Using the formulae, we can obtain the relationship between Bezier and Said-Ball points as show in the following table:

Degree	Bezier points of Said-Ball curve	Said-Ball points of Bezier curve
2	$b_0 = v_0$	$v_0 = b_0$
	$b_1 = v_1$	$v_1 = b_1$
	$b_2 = v_2$	$v_2 = b_2$
3	$b_0 = v_0$	$v_0 = b_0$
	$b_1 = (v_0 + 2v_1)/3$	$v_1 = (3b_1 - b_0)/2$
	$b_2 = (v_3 + 2v_2)/3$	$v_2 = (3b_2 - b_3)/2$
	$b_3 = v_3$	$v_3 = b_3$
4	$b_0 = v_0$	$v_0 = b_0$
	$b_1 = (v_0 + 3v_1)/4$	$v_1 = (-b_0 + 4b_1)/3$
	$b_2 = v_2$	$v_2 = b_2$
	$b_3 = (v_4 + 3v_3)/3$	$v_3 = (4b_3 - b_4)/3$
	$b_4 = v_4$	$v_4 = b_4$

Note: See more relationship in the lecture note.

Bezier Curves v. Said-Ball Curves

The Bezier points for a Said-Ball curve satisfy the following conditions:

$b_0 = v_0$

$b_n = v_n$ $\quad$ n is the degree of the curve

b_k is a convex combination of $\{v_i\}_0^k$, for $i \leq \lfloor (n-1)/2 \rfloor$

b_k is a convex combination of $\{v_i\}_{n-1}^k$, for $i \geq \lfloor (n-1)/2 \rfloor$

- This results ensures that the Said-Ball curve interpolates two end-points.
- The Bezier points lie inside the Said-Ball convex hull formed by the control points of the Said-Ball curve.
- For the same set of control points, the Said-Ball curve is farther from the control polygon than the Bezier curve.

Relationship between Bezier and Wang-Ball Curves

- Recall that we know that Bezier points $\{b_k\}_0^n$ of a Wang-Ball curve of degree n can be obtained in terms of Wang-Ball control points (from Lecture 6).

$$b_j = \sum_{i=0}^{n} A_{i,j}^n p_i$$

where

$$A_{i,j}^n = \begin{cases} 2^i \frac{\binom{n-2-2i}{j-i}}{\binom{n}{j}}, & i < n/2 \\ 2^{n-i} \frac{\binom{2i-2-n}{i-j}}{\binom{n}{j}}, & i > n/2 \\ \frac{2^{\lceil (n-1)/2 \rceil}}{\binom{n}{\lceil (n-1)/2 \rceil}}, & i = j = \lceil n/2 \rceil \text{ or } \lfloor (n+1)/2 \rfloor \\ 0, & \text{otherwise} \end{cases}$$

Relationship, continue

- The proof of the formula makes use of polar form.
- The proof is in the lecture note, if you are interested.
- Now suppose we want to convert a set of Bezier control points to a set of Wang-Ball control points. The transformation matrix is given by

$$[b_0, b_1, \ldots, b_n] = [p_0, p_1, \ldots, p_n] \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,n} \end{bmatrix}$$

Relationship, continue

- What if you want Wang-Ball control points of Bezier curve in terms of Bezier control points?
- This can be done by

$$[p_0, p_1, \ldots, p_n] = [b_0, b_1, \ldots, b_n] \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,n} \end{bmatrix}^{-1}$$

Homework: Review how to find an inverse of a matrix.

Relationship, Table

From the above formulae, relationships between Bezier and Wang-Ball control points can be given as in the following table:

Degree	Bezier CP of a Wang-Ball curve	Wang-Ball CP of a Bezier curve
3	$b_0 = p_0$ $b_1 = (p_0 + 2p_1)/3$ $b_2 = (2p_2 + p_3)/3$ $b_3 = p_3$	$p_0 = b_0$ $p_1 = (3b_1 - b_0)/2$ $p2 = (3b_2 - b_3)/2$ $p_3 = b_3$
4	$b_0 = p_0$ $b_1 = (p_0 + p_1)/2$ $b_2 = (p_0 + 4p_2 + p_4)/6$ $b_3 = (p_3 + p_4)/2$ $b_4 = p_4$	$p_0 = b_0$ $p_1 = -b_0 + 2b_1$ $p_2 = (-b_0 + 6b_2 - b_4)/4$ $p_3 = 2b_3 - b_4$ $p_4 = b_4$

Note: More can be found in the lecture note.

Analysis of Algorithms: Bezier curve

Recall that we have seen the time complexity of de Casteljau Algorithm.

- The algorithm requires $n(n+1)/2$ additions and $n(n+1)$ multiplications.
- The algorithm is $O(n^2)$.

Wang-Ball curve

- We have seen the recursive algorithm to compute a point on Wang-Ball curve.
- That algorithm is called Wang Algorithm.
- The algorithm requires $(3n-1)/2$ additions and $(3n-1)$ multiplications for n is odd and $3n/2$ additions and $3n$ multiplications for n is even.
- This show that the Wang Algorithm is $O(n)$ algorithm.

Surface Modeling

Definition: In mathematics, surface is a two-dimensional topology manifold.

- The most familiar surface are boundary of solid objects in $\mathbb{R}^3$.
- We will discuss surface that can be written as a vector function of two parameters, say u and v.
- The function will maps a region D of the uv-plane into $\mathbb{R}^3$.
- That is a mapping of $\mathbb{R}^2 \to \mathbb{R}^3$ or $(u, v) \to (x, y, z)$.
- Comparing to what we have seen so far is $t \to (x, y)$ or $\mathbb{R} \to \mathbb{R}^2$.
- Surface can also be considered as a vector-value function of two parameters u and v.

$$S(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$$

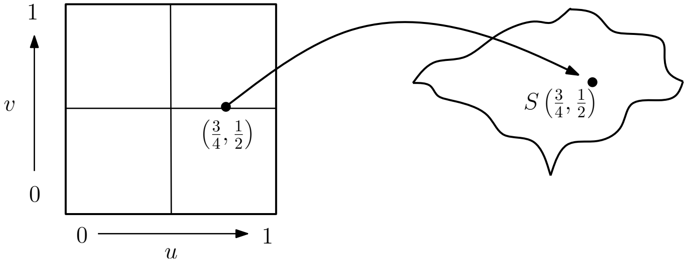
where $0 \leq u, v \leq 1$.

Representing Surface

- There are many different ways to represent surface.
- They are different in the coordinate functions used and the type of region D.
- The simplest and most widely used method is tensor product scheme.
- The tensor product scheme is basically a bidirectional curve scheme.
- It still uses basis functions and geometric coefficients.
- However the basis function is now defined as a function of u and v.

Tensor Product

- Since tensor product is bidirectional curve scheme. This means we consider two ways simultaneously.
- For example $\left(\frac{3}{4}, \frac{1}{2}\right) \rightarrow S\left(\frac{3}{4}, \frac{1}{2}\right) \in \mathbb{R}^3$



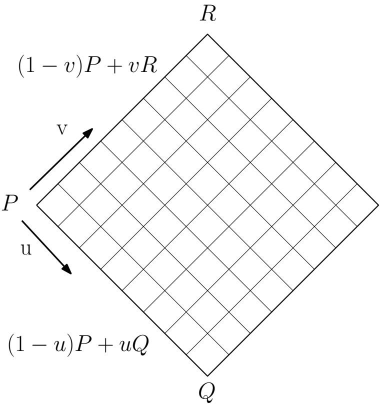
- The simplest surface is a plane.

A plane

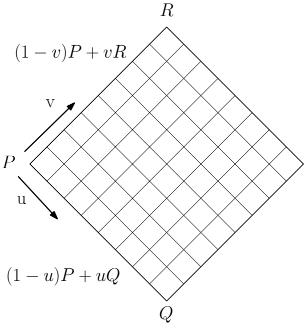
- The general equation of a plane is

$$P(x, y, z) \;:\; Ax + By + Cz + D = 0$$

- A sweeping of a line segment into another direction of another line segment:



A plane, continue



Let $x(u)$ be a curve, express linearly in term of the basis function $F_i(u)$

$$x(u) = \sum_{i=0}^{m} c_i F_i(u)$$

$$c_i(u) = \sum_{j=0}^{n} P_{i,j} G_j(v)$$

If we sweep each control point c_i, the resulting is called the tensor product surface as given by

A plane, continue

If we sweep each control point c_i, the resulting is called the tensor product surface as given by

$$
\begin{aligned}
S(u, v) &= \sum_{i=0}^{m} c_i F_i(u) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} G_j(v) F_i(u)
\end{aligned}
$$

where F_i and G_j are any basis function.

Tensor Product of a Plane

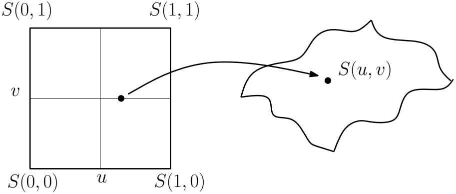
- Let $\{P_{i,j}\}$ be the control point for the case of the curve.
- However for the case of the surface, we call $\{P_{i,j}\}$ the control net.
- This consists of $(m+1) \times (n+1)$ control points then the coordinate of a point $S(u,v)$ can be computed by two step procedure:

1. $c_i(v) = \sum_{j=0}^{n} P_{i,j} G_j(v)$
2. $S(u,v) = \sum_{i=0}^{m} c_i(v) F_i(u)$

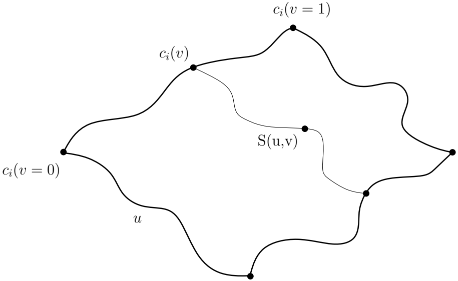
Finding a Point on a Surface

Two ways to calculate any particular point on a surface:

1. Directly compute a point using $S(u, v)$



2. First calculate the point in one direction, then map that point to another direction.



Surface in Power Basis

We can consider a surface in power basis that is given by

$$S(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j} u^i v^j$$
$$= \left[u^i\right]^T [a_{i,j}] \left[v^j\right]$$

where $0 \leq u, v \leq 1$.

- $\left[u^i\right]^T$ is a $1 \times (m+1)$ row vector
- $\left[v^j\right]$ is a $(n+1) \times 1$ column vector
- $[a_{i,j}]$ is a $(m+1) \times (n+1)$ matrix of three dimensional points.

In case of cubic,

$$\begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} a_{i,j} \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

Surface in Power Basis, continue

$$
\begin{aligned}
S(u,v) &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} u^i v^j \\
&= \left[u^i\right]^T \left[a_{i,j}\right] \left[v^j\right]
\end{aligned}
$$

where $0 \leq u, v \leq 1$.

- $\left[u^i\right]^T$ is a $1 \times (m+1)$ row vector
- $\left[v^j\right]$ is a $(n+1) \times 1$ column vector
- $[a_{i,j}]$ is a $(m+1) \times (n+1)$ matrix of three dimensional points.

This power basis is similar to the non-power basis above

$$S(u,v) = [f_i(u)]^T [P_{i,j}] [G_j(v)]$$

Recap

- Relationship between Bezier and Said-Ball points
- Relationship between Bezier and Wang-Ball points
- Computation aspect

Surface Model

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Next time we will see how to define Bezier surface.