Lecture 7

178 359
Simulation and Modeling

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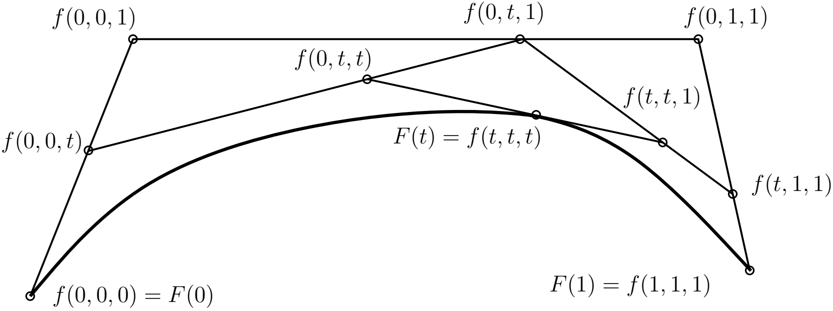
Overview

Polar Form Approach:

- Simple Illustration of the polar form representation
- Properties related to polar form approach
- Polar form principle
- Obtaining polar form
- Polar form for Bezier curve

Simple Illustration

The main idea behind polar forms is best explained by a picture.



Polar Form Approach

- In the picture, it is just a standard de Casteljau Algorithm.
- The difference is the label scheme; it is being used in the form $f(\cdot,\cdot,\cdot)$ where f is the polar form of the polynomial F.
- Since F is of degree three, its polar form has three arguments.
- Polar form is symmetric, i.e. its three arguments can be written in any order without changing the value of f.
- f is related to F by the identity $F(u) = f(u, u, u)$. (u is our old t, but $u \in [r, s]$).

Polar Form Approach cont.

- The incidence structure of the points and lines in previous figure is reflected in the labels: All points whose labels share at least two arguments lie on the same line.
- As t changing value from 0 to 1, the point is moving away from the starting point to the other end point.
- This means the polar form is affine in each argument.
- And because we have multiple arguments in our affine form, we shall call the affine form simply multiaffine.
- Thus the polar form f of a cubic polynomial curve F is a symmetric triaffine map that satisfies $F(u) = f(u, u, u)$.

How To Find a Point with Polar Form

From the previous example (given the control points $f(0,0,0)$, $f(0,0,1)$, $f(0,1,1)$ and $f(1,1,1)$), finding a point on the curve in polar form representation can be described as follows:

- Interpolate linearly along the edges of the control polygon to obtain the points $f(0,0,t)$, $f(0,t,1)$ and $f(t,1,1)$.
- Then interpolate linearly between these points to obtain $f(0,t,t)$ and $f(t,t,1)$.
- Finally interpolate between these two points to obtain the point $F(t) = f(t,t,t)$.
- This is exactly the de Casteljau Algorithm.

Properties:

Before we generalize the polar form representation, let's review some properties related to the form:

Affine Combination

- Let $\{x_i\} \in \mathbb{R}^d$, a linear combination, $\sum a_i x_i, a_i \in \mathbb{R}$ is affine if $\sum a_i = 1$.
- In the case of affine form $f : \mathbb{R} \rightarrow \mathbb{R}^d$, f is affine if it preserves affine combination, i.e. if f satisfies $f\left(\sum_j a_j u_j\right) = \sum_j a_j f(u_j)$ for all a_j with $\sum a_j = 1$.

Affine Space

- $U \subset \mathbb{R}^d$ is an affine space if it is closed under any affine combinations, i.e. $\forall x_i \in U,\ a_i \in \mathbb{R}$ and $\sum a_i = 1$ then $\sum a_i x_i \in U$.
- Note that an affine space is like a linear space but it lacks a distinguished origin.
- Choose an origin for the affine space U, then the difference between the two points is a vector and the set of these vectors is a vector space U^*.

Affine Space, Example

Example: Choose x_1 as the origin, then a point $u \in U$ can be written as:

$$u = \sum_{i=1}^{n} a_i x_i = x_1 + \sum_{i=1}^{n} a_i (x_i - x_1)$$

$$u - x_1 = \sum_{i=1}^{n} a_i (x_i - x_1)$$

$$\overrightarrow{u} = \sum_{i=1}^{n} a_i \overrightarrow{x}_i \quad \text{with} \quad \overrightarrow{x_i} = x_i - x_1, \overrightarrow{u} = u - x_1$$

Affine Frame

$\{x_i\}_1^k \in U$ is an affine frame of the affine space U if:

- $\{x_i\}$ is affinely independent i.e. no x_i can be expressed as an affine combination of others.
- $\{x_i\}$ spans U, i.e. $\forall x \in U$, it can be expressed as an affine combination of $\{x_i\}$

If k is the number of elements in an affine frame then $k-1$ is the dimension of the space. An affine frame is an analog of a linear basis with the same dimension of the vector space.

Theorem

Theorem: If we choose an element of an affine frame as origin for the affine space, for example x_1, then $\{\overrightarrow{x_i} = x_i - x_1\}_2^k$ form a basis of the linear space U^*.

Proof: See the Lecture note in Chapter 2.

Affine Function

- Let P and O be two affine spaces. A function $F : P \rightarrow O$ is affine if it preserves affine combinations, i.e.:

$$\forall x_i \in P, \forall a_i \in \mathbb{R}, \sum a_i = 1 \text{ then } F(\sum a_i x_i) = \sum a_i F(x_i)$$

- F is affine iff when we choose the origins and bases for the domain and codomain, each coordinate of $F(x)$ can be written as a polynomial of degree at most 1 in the coordinates of x.
- This is a properties that all the curve models we have seen hold.

Multiaffine Function

- A function $F : P^n \to O$ is multiaffine if it is affine in its arguments:

$$F(u_1, \ldots, \sum_j a_j u_{ij}, \ldots, u_n) = \sum_j a_j F(u_1, \ldots, u_{ij}, \ldots, u_n)$$

- for all $a_j \in \mathbb{R}$ with $\sum_j a_j = 1$
- F is multiaffine iff after choosing the origins and bases for P and O, each coordinate of $F(u_1, \ldots, u_n)$ is a polynomial of degree at most 1 in the coordinates of $\{u_i\}$.
- This is the property of polar form.

Symmetric Function

- $F : P^n \rightarrow O$ is symmetric if the value of F remains unchanged when we make any interchange of their arguments:

$$F(\pi(u_1, \dots, u_n)) = F(u_1, \dots, u_n)$$

  where $\pi(u_1, \dots, u_n)$ is any permutation of $u_1, \dots, u_n$
- This is also a property of polar form.

Polar Form Principle (so-called Blossoming Principle)

We are now like to establish the Polar Form Principle (so-called Blossoming Principle; Ramshaw 1988)

Theorem: Polynomials $F : \mathbb{R} \to \mathbb{R}^d$ of degree n and symmetric multiaffine maps $f : \mathbb{R}^n \to \mathbb{R}^d$ are equivalent to each other. In particular, given a map of either type exists that satisfies the identity $F(u) = f(u, \ldots, u)$. In this situation f is called the multiaffine polar form of blossom of F, while F is called the diagonal of f. Furthermore, the $q - th$ derivative of F is given as

$$F^{(q)}(u) = \frac{n!}{(n-q)!} f(\underbrace{u, \ldots, u, \widehat{1}}_{n-q}, \underbrace{\ldots, \widehat{1}}_{q})$$

where $\widehat{1} = 1 - 0 \in \mathbb{R}$ is the standard unit vector and $f(u, \ldots, u, \widehat{1}, \ldots, \widehat{1})$ is defined as above.

Polar Form Principle cont.

The theorem essentially states that each polynomial has a unique polar form.

- This implies that polar form function is a one-to-one function.
- Without this property, polar form wouldn't be useable.

Definition of Polar Form and Diagonal Form

- If $F(u)$ is a polynomial of degree (at most) n, the polar form of F (n-polar form of F) is the unique, symmetric, n-affine function $f(u_1, \ldots, u_n)$ that satisfies $f(u, \ldots, u) = F(u)$.
- In this case, F is called the diagonal form of f, $F(u)$ is called the diagonal value of f and parameter u is called the diagonal argument of F. Moreover, the value $f(u_1, \ldots, u_n)$ is called the polar value of F and each u_i is a polar argument.
- Note: To prevent any confusion, we shall denote u^n as the list of $\underbrace{u, \ldots, u}_{n}$.

Methods to obtain the polar form for a polynomial

There are a number of methods to obtain the polar form for a polynomial such as:

- using differential geometry
- using the elementary symmetric functions
- using probability theory
- using the Taylor series of an auxiliary function
- using integration by parts.

We will discuss some of these methods.

Elementary Symmetric Functions

- The elementary symmetric function and probability theory methods replace t^i by $\sigma_{i,n}(t_1, \ldots, t_n)$.
- $\sigma_{i,n}(t_1, \ldots, t_n)$ denotes the average value of all products consisting of i elements of $t_1, \ldots, t_n$.
- For example, the polar form of degree 3 ($3 - \textit{polarform}$) of the polynomial $F(t) = t^3 + 3t^2 + 5t + 1$ can be obtained by replacing

$$\begin{aligned} t^3 &\text{ by } \sigma_{3,3}(t_1, t_2, t_3) = t_1 t_2 t_3 \\ t^2 &\text{ by } \sigma_{2,3}(t_1, t_2, t_3) = (t_1 t_2 + t_1 t_3 + t_2 t_3)/3 \\ t &\text{ by } \sigma_{1,3}(t_1, t_2, t_3) = (t_1 + t_2 + t_3)/3 \end{aligned}$$

This gives

$$f(t_1, t_2, t_3) = t_1 t_2 t_3 + (t_1 t_2 + t_1 t_3 + t_2 t_3) + \frac{5}{3}(t_1 + t_2 + t_3) + 1$$

Taylor Series

- A curve $F(t)$ can be written as a Taylor series associate with τ as

$$F(t) = \sum_{i=0}^{\infty} \frac{F^{(i)}(\tau)(t-\tau)^i}{i!}$$

- If $F(t)$ is of degree n then we have

$$F(t) = \sum_{i=0}^{n} \frac{F^{(i)}(\tau)(1-\tau)}{i!}$$

The polar form of $(1-\tau)^i$ can be represented as

$$\frac{i!}{n!}(-D)^{n-i} \prod_{1}^{n} (t_i - \tau)$$

where D denotes the differential in τ.

Taylor Series cont.

$$\Phi_i(\tau) = \prod_1^n (t_i - \tau)$$

is the auxiliary function. Therefore the polar form of F is

$$f(t_1, \ldots, t_n) = \sum_{i=0}^{n} \Phi^{(n-i)}(\tau) F^{(i)}(\tau)$$

Finding $n+1$ degree polar form from n degree polar form

- The polar form of polynomial degree $(n+1)$ can be derived from its polar form of the polynomial of degree n.
- This can be done by taking the average of all $n+1$ polar values $f(t_{i_1},\ldots,t_{i_n})$ where $t_{i_1},\ldots,t_{i_n}$ are any combinations consisting of n arguments from $\{t_i\}_1^{n+1}$. That is

$$f(t_1,\ldots,t_{n+1}) = \frac{1}{n+1}\left(f(t_1,\ldots,t_n) + f(t_2,\ldots,t_{n+1}) + \ldots \right.$$
$$\left. + f(t_{n+1},t_1,\ldots,t_{n-1})\right)$$

Polar Form of Multi Polynomials

- The polar form of $\sum F_i$ is the sum of their polar forms at largest degree. It can also be obtained by symmetrizing the sum of their polar forms.
- The polar form of F times G can be obtained by symmetrizing the product of their polar forms with distinct arguments, i.e. renaming all arguments of g distinct from f, making the product and then symmetrizing the result.
- The polar form of $F\ o\ G$ can be obtained by renaming arguments of g each time replacing arguments of f by g.

Example

Example: Given $F(t) = t^2 + 2t$ and $G(t) = t^3 + 1$ and $f(t_1, t_2) = t_1 t_2 + t_1 + t_2$, $g(t_1, t_2, t_3) = t_1 t_2 t_3 + 1$.

- $F + G = t^3 + t^2 + 2t + 1$
- $FG = t^5 + 2t^4 + t^2 + 2t$
- $F \ o \ G = t^6 + 4t^3 + 3$

Example cont.

- $(f+g)(t_1,t_2,t_3) = t_1t_2t_3 + t_1t_2 + t_1 + t_2 + 1$. Symmetrizing $f+g$ as

$$\frac{1}{6}\left[(f+g)(t_1,t_2,t_3) + (f+g)(t_1,t_3,t_2) + (f+g)(t_2,t_1,t_3)\right.$$
$$\left.+(f+g)(t_2,t_3,t_1) + (f+g)(t_3,t_1,t_2) + (f+g)(t_3,t_2,t_1)\right]$$
$$= t_1t_2t_3 + \frac{1}{3}(t_1t_2 + t_1t_3 + t_2t_3) + \frac{2}{3}(t_1+t_2+t_3) + 1$$

- This is the polar form of $F+G$. We can also obtain the polar form for $F+G$ by increasing the degree of f as:

$$\begin{aligned} f^*(t_1,t_2,t_3) &= \frac{f(t_1,t_2) + f(t_1,t_3) + f(t_2,t_3)}{3} \\ &= \frac{t_1t_2 + t_1t_3 + t_2t_3}{3} + \frac{2}{3}(t_1+t_2+t_3) \end{aligned}$$

- By taking the sum of f and g we obtain the same result.

Example cont.

- Renaming arguments of g distinct from f and making the product fg

$$f(t_1, t_2)g(t_3, t_4, t_5) = t_1 t_2 t_3 t_4 t_5 + t_1 t_3 t_4 t_5 + t_2 t_3 t_4 t_5 + t_1 t_2 + t_1 + t_2$$

- Symmetrizing fg we obtain:

$$t_1 t_2 t_3 t_4 t_5 + \frac{2}{5}(t_1 t_2 t_3 t_4 + t_1 t_2 t_3 t_5 + t_1 t_2 t_4 t_5 + t_1 t_3 t_4 t_5 + t_2 t_3 t_4 t_5)$$
$$+ \frac{1}{10}(t_1 t_2 + t_1 t_3 + t_1 t_4 + t_1 t_5 + t_2 t_3 + t_2 t_4 + t_2 t_5 + t_3 t_4 + t_3 t_5$$
$$+ t_4 t_5) + \frac{2}{5}(t_1 + t_2 + t_3 + t_4 + t_5)$$

- This is the polar form of FG.

Example cont.

- Substituting all arguments of f by g whose all arguments are reparameterized (distinct from the arguments of g at the previous times) we obtain:

$$f(g(t_1, t_2, t_3), g(t_4, t_5, t_6)) = t_1 t_2 t_3 t_4 t_5 t_6 + t_1 t_2 t_3 + t_4 t_5 t_6 + 1$$

- Symmetrizing the right hand side of the above, we obtain:

$$t_1 t_2 t_3 t_4 t_5 t_6 + \frac{2}{10}(t_1 t_2 t_3 + t_1 t_2 t_4 + \ldots + t_4 t_5 t_6) + 1$$

- This is the polar form of $F\ o\ G$.

Polar Form for Bezier Curves

Theorem: The polar form of a Bezier curve of degree n can be given by

$$f(t_1, \dots, t_n) = \sum_{i=0}^{n} \binom{n}{i} \sigma^{i}_{i,n-i}(t_1, \dots, t_n)$$

where $\sigma^{i}_{i,n-i}(t_1, \dots, t_n)$ is the mean of all the products of the form $t_{j1} \cdots t_{ji}(1 - t_{ji+1}) \cdots (1 - t_{jn})$.

Proof: This is not very difficult. See the lecture note for explanation.

Polar Form for Bezier Curves, cont.

Theorem: Let $\{b_0, \ldots, b_n\}$ be the control points of a Bezier curve defined over $[0, 1]$. The polar form function f of this curve satisfies the following conditions:

$$f(0^{n-i}1^i) = b_i \text{ for } i = 0, \ldots, n$$

where 0^{n-i} denotes $\underbrace{0, \ldots, 0}_{n-i}$ and 1^i denotes $\underbrace{1, \ldots, 1}_{i}$

Polar Form for Bezier Curves, Example

Example: Consider a Bezier curve of degree 3 which has control points b_0, b_1, b_2 and b_3. Find polar form of these control points.

Solution:

$$
\begin{aligned}
b_0 &= f(0^3 1^0) &&= f(0,0,0) \\
b_1 &= f(0^2 1^1) &&= f(0,0,1) \\
b_2 &= f(0^1 1^2) &&= f(0,1,1) \\
b_3 &= f(0^0 1^3) &&= f(1,1,1)
\end{aligned}
$$

Recap

We have been talking about:

- Polar form approach
- Definition
- Simple illustration
- Blossom principle
- Methods to find polar form
- Polar form for Bezier curve

Next lecture, we shall revisit the relationship between curves and talk a little bit on Rational curve.