Lecture 6

178 359
Simulation and Modeling

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Overview

Review of Bezier curve

- Derivative of Bezier curve
- Degree elevation

Bezier Curve Modeling (cont.)

- Rational Bezier Curve

Ball curve modeling

- Ball curve
- Said-Ball curve
- Wang-Ball curve

New Recursive Algorithm for Wang-Ball Curve

- The point $\mathbf{W}(t)$ on the Wang-Ball curve corresponding to the parametric value t can also be recursively defined as follows:

$$p_i^r(t) \begin{cases} p_i, & i < \left\lfloor \frac{n-r}{2} \right\rfloor \\ p_{i+1}, & i > \left\lceil \frac{n-r}{2} \right\rceil \\ (1-t)p_i^{r-1} + tp_{i+1}^{r-1}, & \text{otherwise} \end{cases}$$

where $p_i^0(t) = p_i$, $r = 1, 2, \ldots, n$ and $i = 0, 1, \ldots, n-r$ and $\mathbf{W}(t) = p_0^n(t)$.

This algorithm runs with $\left\lceil \frac{3n-1}{2} \right\rceil$ additions and $2\left\lceil \frac{3n-1}{2} \right\rceil$ multiplications.

Degree Elevation

- Given control points $\{v_i\}$, $i = 0, 1, \ldots, n$, the Wang-Ball curves of degree n can be expressed in terms of the Wang-Ball basis of degree $n+1$ (degree elevation) as

$$\mathbf{W}(t) = \sum_{i=0}^{n+1} p_i^{(1)} A_i^{n+1}$$

$$p_i^{(1)} = \begin{cases} p_i, & 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \\ \frac{p_{i+1}+p_i}{2}, & i = \left\lceil \frac{n+1}{2} \right\rceil \\ p_i, & = \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq i \leq n+1 \end{cases}$$

where p_i is a control point of the degree n Wang-Ball curve, and $p_i^{(1)}$ is a control point of the degree $n+1$ Wang-Ball curve.

Hermite Curve VS Berzier Curve

A Hermite curve can be considered as a cubic Bezier curve. Recall that the Hermite curve can be represented by the first and the last control points p_0 and p_3 and two tangent lines at those control points R_0 and R_3. And recall that a cubic Bezier curve are

$$\mathbf{B}(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3$$
$$\mathbf{B}'(t) = (-3t^2+6t-3)p_0 + (9t^2-12t+3)p_1 + (-9t^2+6t)p_2 + (3t^2)p_3$$

Two endpoints:

$$\mathbf{B}(0) = p_0 \quad \text{and} \quad \mathbf{B}(1) = p_3$$

Two tangents:

$$\mathbf{B}'(0) = 3(p_1 - p_0) = R_0$$
$$\mathbf{B}'(1) = 3(p_3 - p_2) = R_1$$

Hermite Curve VS Ball Curve

A Hermite curve can be considered as a Ball curve. Recall that Ball curve parametric equation is

$$\beta(t) = (1-t)^2 p_0 + 2t(1-t)^2 p_1 + 2t^2(1-t)p_2 + t^2 p_3$$
$$\beta'(t) = (2t-2)p_0 + (6t^2 - 8t + 2)p_1 + (-6t^2 + 4t)p_2 + (2t)p_3$$

Two endpoints:

$$\beta(0) = p_0 \quad \text{and} \quad \beta(1) = p_3$$

Two tangents:

$$\beta'(0) = 2(p_1 - p_0) = R_0$$
$$\beta'(1) = 2(p_3 - p_2) = R_1$$

Cubic Bezier Curve VS Ball Curve

A cubic Bezier curve can be considered as a Ball curve

$$\mathbf{B}(t) = (1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_2 + t^3 b_3$$

$$\beta(t) = (1-t)^2 v_0 + 2t(1-t)^2 v_1 + 2t^2(1-t) v_2 + t^2 v_3$$

Given a Ball curve, it can be expressed in terms of a Bezier curve by equating to equations

$$(1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_2 + t^3 b_3 = \\ (1-t)^2 v_0 + 2t(1-t)^2 v_1 + 2t^2(1-t) v_2 + t^2 v_3$$

Cubic Bezier Curve VS Ball Curve: 2

From

$$(1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_2 + t^3 b_3 = \\ (1-t)^2 v_0 + 2t(1-t)^2 v_1 + 2t^2(1-t) v_2 + t^2 v_3$$

Expanding and then rearranging in term of t:

$$b_0 + (-3b_0 + 3b_1)t + (3b_0 - 6b_1 + 3b_2)t^2 \\ + (-b_0 + 3b_1 - 3b_2 + b_3)t^3 \\ = v_0 + (- 2v_0 + 2v_1)t + \\ (v_0 - 4v_1 + 2v_2 + v_3)t^2 + (2v_1 - 2v_2)t^3$$

Cubic Bezier Curve VS Ball Curve: 3

Considering each term of t:

$$
\begin{aligned}
b_0 &= v_0 \\
-3b_0 + 3b_1 &= -2v_0 + 2v_1 \\
3b_0 - 6b_1 + 3b_2 &= v_0 - 4v_1 + 2v_2 + v_3 \\
-b_0 + 3b_1 - 3b_2 + b_3 &= 2v_1 - 2v_2
\end{aligned}
$$

$$
\begin{aligned}
-3b_0 + 3b_1 &= -2v_0 + 2v_1 \\
-3v_0 + 3b_1 &= -2v_0 + 2v_1 \\
3b_1 &= -2v_0 + 2v_1 + 3v_0 \\
b_1 &= \frac{1}{3}v_0 + \frac{2}{3}v_1
\end{aligned}
$$

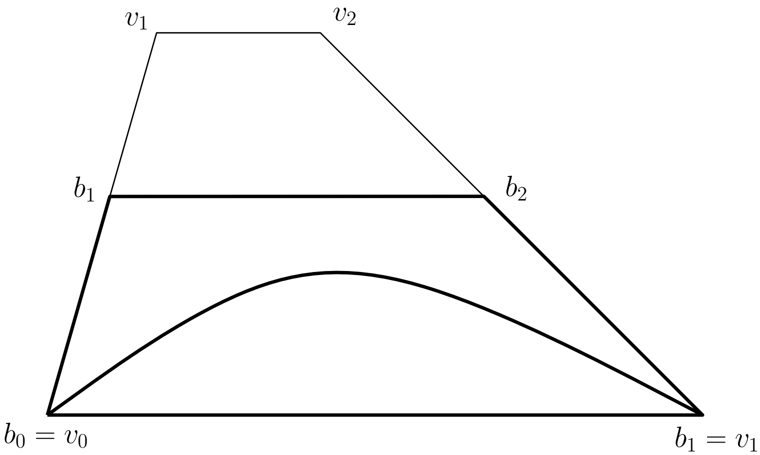
Cubic Bezier VS Ball Curve: 4

Consider each term of t

$$3b_0 - 6b_1 + 3b_2 = v_0 - 4v_1 + 2v_2 + v_3$$
$$3v_0 - 2v_0 - 4v_1 + 3b_2 = v_0 - 4v_1 + 2v_2 + v_3$$
$$3b_2 = 2v_2 + v_3$$
$$b_2 = \frac{2}{3}v_2 + \frac{1}{3}v_3$$

$$-b_0 + 3b_1 - 3b_2 + b_3 = 2v_1 - 2v_2$$
$$-v_0 + v_0 + 2v_1 - 2v_2 - v_3 + b_3 = 2v_1 + 2v_2$$
$$-v_3 + b_3 = 0$$
$$b_3 = v_3$$

Cubic Bezier VS Ball Curve: Illustration



Cubic Bezier VS Ball Curve: Summarize

$$
\begin{aligned}
b_0 &= 1v_0 + 0v_1 + 0v_2 + 0v_3 \\
b_1 &= \frac{1}{3}v_0 + \frac{2}{3}v_1 + 0v_2 + 0v_3 \\
b_2 &= 0v_0 + 0v_1 + \frac{2}{3}v_2 + \frac{1}{3}v_3 \\
b_3 &= 0v_0 + 0v_1 + 0v_2 + 1v_3
\end{aligned}
$$

Cubic Bezier VS Ball Curve: Note

Remarks: (important):

- Each Bezier control point is a convex combination of the Ball control points.
- The Bezier control points lie in the convex hull of the Ball control points.
- The Bezier control polygon approximates the curve better than Ball's.
- The Bezier control polygon is closer to the curve.

Cubic Bezier VS Ball Curve: Matrix Represenation

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Relationship between Bezier and Said-Ball Curves

The Bezier point of a Said-Ball curve of degree n can be given in terms of Said-Ball control points by

$$b_j = \sum_{i=0}^{n} S_{i,j}^{n} v_i$$

where

$$S_{i,j}^{n} = \begin{cases} \dfrac{\binom{\lfloor n/2 \rfloor + i}{i}\binom{n-\lfloor n/2 \rfloor - i - 1}{j-i}}{\binom{n}{j}}, & i < n/2 \\[2ex] \dfrac{\binom{n+\lfloor n/2 \rfloor - i}{n-i}\binom{n-\lfloor n/2 \rfloor - i - 1}{n-j-i}}{\binom{n}{j}}, & i > n/2 \\[2ex] 1, & i = j = n/2 \\[2ex] 0, & \text{otherwise} \end{cases}$$

Relationship Between Bezier and Wang-Ball Curve

The Bezier point of a Wang-Ball curve of degree n can be given in terms of Wang-Ball control points by

$$b_j = \sum_{i=0}^{n} A_{i,j}^n p_i$$

where

$$A_{i,j}^n = \begin{cases} 2^i \frac{\binom{n-2-2i}{j-i}}{\binom{n}{j}}, & i < n/2 \\ 2^{n-i} \frac{\binom{2i-2-n}{i-j}}{\binom{n}{j}}, & i > n/2 \\ \frac{2^j}{\binom{n}{j}}, & i = j = n/2 \\ \frac{2^{n-j}}{\binom{n}{j}}, & i = j = n/2 \\ 0, & \text{otherwise} \end{cases}$$