Lecture 5

178 359
Simulation and Modeling

Pattarawit Polpinit

Department of Computer Engineering
Khon Kaen Uiversity

Overview

Review of Bezier curve

- Derivative of Bezier curve
- Degree elevation

Bezier Curve Modeling (cont.)

- Rational Bezier Curve

Ball curve modeling

- Ball curve
- Said-Ball curve
- Wang-Ball curve

Rational Bezier curves

- Sometimes we want to provide a closer approximation to some arbitrary shape.
- **Rational Bezier curve** provides us with that tool. We add adjustable weights to Bezier curve.

$$\mathbf{B}(t) = \frac{\sum_{i=0}^{n} B_n^i(t) b_i w_i}{\sum_{i=0}^{n} B_n^i(t) w_i}$$

  for $i \in [0, 1]$ and $w_i > 0$.
- The w_i are scalar number that is called weight.
- Hence $\mathbf{B}(t)$ has the same common denominator.

Ball Curve

The cubic Ball curve was defined by Ball in 1974 as

$$\mathcal{B}(t) = \sum_{i=0}^{3} v_i \cdot \beta_i(t), \quad t \in [0, 1]$$

It uses different basis functions $\beta_i(t)$ from Bezier curve to define the curve.

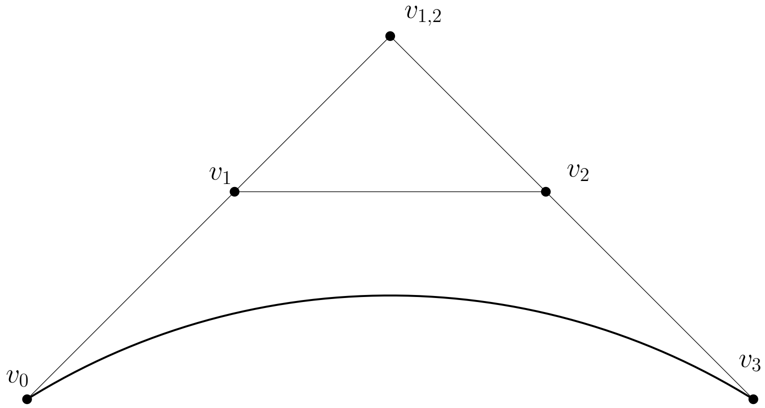
$$\begin{aligned}
\beta_0(t) &= (1-t)^2 \\
\beta_1(t) &= 2t(1-t)^2 \\
\beta_2(t) &= 2t^2(1-t) \\
\beta_3(t) &= t^2
\end{aligned}$$

Cubic Ball Curve: Properties

1. Coefficients are nonnegative. $\beta_i \geq 0$ for $i = \{0, 1, 2, 3\}$
2. Affine combination $\sum_{i=0}^{3} \beta_i(t) = 1$
3. Symmetry with respect to t, $\beta_i(t) = \beta_{3-i}(1-t)$.
4. There is a special case that reduces the curve to a quadratic curve if the two middle point coincide.

$$\mathcal{B}(t) = \sum_{i=0}^{3} v_i \cdot \beta_i(t), \quad t \in [0, 1]$$

Cubic Ball Curve: Properties 2



Matrix Representation of Cubic Ball Curve

$$\mathcal{B}(t) = G \cdot M \cdot T$$

where

$$G = [\ v_0 \quad v_1 \quad v_2 \quad v_3\ ]$$

$$M = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 2 & -4 & 2 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Generalized Ball Curve

There have been two models that generalizes Ball curve.

1. Said-Ball curve: 1989
2. Wang-Ball curve: 1987

Said-Ball curve

Said presented his work in 1989. Given $n+1$ control points, his curve is defined as follows:

$$\mathbf{C}(t) = \sum_{i=0}^{2m+1} v_i S_i^{2m+1}(t), \qquad 0 \le t \le 1$$

where

- $S_i^{2m+1}(t) = \binom{m+i}{m} t^i (1-t)^{m+1}$, if $i \le m$
- $S_i^{2m+1}(t) = S_{2m+1-i}^{2m+1}(1-t)$, if $i \ge m+1$

for any integer $m \ge 1$ and given a set of $(2m+2)$ points $\{v_i\}_0^{2m+1}$.

This implies that $n = 2m+1$.

Note: The curve is defined only for odd degree.

Said-Ball Curve: Properties

1. Nonnegative basis functions: $S_i^{2m+1}(t) \geq 0$
2. Partition of unity: $\sum_{i=0}^{2m+1} S_i^{2m+1}(t) = 1$
3. Convex hull property: The Said-ball curve lies inside the convex hull formed by its control points.
4. The Said-Ball curve interpolates two end-points: because $S_0^{2m+1}(0) = 1$ and $S_{2m+1}^{2m+1}(1) = 1$.

Degree of Said-Ball curves

- If two control points v_m and v_{m+1} are distinct, then the Said-Ball curve is of degree $2m + 1$ or odd degree.
- The degree of the Said-Ball curve will be at most $2m$ (of even degree) when the two points v_m and v_{m+1} coincide.
- This allows us to defines Said-Ball curve of even degree as:

$$\mathbf{C}(t) = \sum_{i=0}^{2m} v_i S_i^{2m}(t), \qquad 0 \leq t \leq 1$$

Degree of Said-Ball curves 2

where $S_i^{2m}(t)$ is defined as follows:

$$
\begin{aligned}
S_i^{2m}(t) &= \binom{m+i}{m} t^i (1-t)^{m+1}, \qquad \text{for } i < m \\
&= \binom{2m}{m} t^m (1-t)^m, \qquad \text{for } i = m \\
&= S_{2m-1}^{2m}(1-t), \qquad \text{for } i \geq m+1
\end{aligned}
$$

Example: Reducing Degree 3 to 2

$$\mathbf{C}(t) = (1-t)^2 v_0 + 2t(1-t)^2 v_1 + 2t^2(1-t)v_2 + t^2 v_3$$

That is the control points v_1 and v_2 are coincide into a new point v_1'.

$$\begin{aligned}
\mathbf{C}(t) &= (1-t)^2 v_0 + \left(2t(1-t)^2 v_1 + 2t^2(1-t)v_2\right) + t^2 v_3 \\
&= (1-t)^2 v_0 + \left(2t(1-t)^2 v_1' + 2t^2(1-t)v_1'\right) + t^2 v_3 \\
&= (1-t)^2 v_0 + \left(2t(1-t)^2 + 2t^2(1-t)\right) v_1' + t^2 v_3 \\
&= (1-t)^2 v_0 + 2t(1-t)\left(t + (1-t)\right) v_1' + t^2 v_3 \\
&= (1-t)^2 v_0 + 2t(1-t)v_1' + t^2 v_3
\end{aligned}$$

Said-Ball Curve: Equation for both odd and even degree

Given $n+1$ control points a Said-Ball curve of degree n defined (for both odd and even degree) by these points, can be expressed as

$$\mathbf{C}(t) = \sum_{i=0}^{n} v_i S_i^n(t)$$

where $S_i^n(t) =$

- $\binom{\lfloor \frac{n}{2} \rfloor + i}{i} t^i (1-t)^{\lfloor \frac{n}{2} \rfloor + 1}, \qquad 0 \leq i \leq \lceil n/2 \rceil - 1$
- $\binom{n}{n/2} t^{n/2} (1-t)^{n/2}, \qquad i = n/2$
- $S_{n-i}^n(1-t), \quad \lfloor n/2 \rfloor + 1 \leq i \leq n$

Said-Ball Curve: Recursive Algorithm

Said developed a recursive algorithm to compute the coordinates of a point on the curve.

The algorithm consists of two steps:

1. Reduce the generalized Said-Ball curve from degree $2m+1$ to a Bezier form of degree $m+1$.
2. Apply the de Calteljau algorithm to this Bezier form to find the coordinate of the intermediate points.

Said-Ball Curve: Wang Algorithm

Wang (1987) developed a recursive algorithm to compute the coordinates of a point on a Wang-Ball curve, bu can also be used to computed the Said-Ball curve. The method was written in Chinese hence no one knew about this until the work of Hu(1996)

To compute the point on the curve corresponding to a special value t on the curve the following algorithm can be used.

- If n is odd: $\widehat{v_i} =$
  - $v_i, \qquad 0 \leq i \leq (n-3)/2$
  - $(1-t)v_{((n-1)/2)} + tV_{(n+1)/2}, \qquad i = (n-1)/2$
  - $v_{i+1}, \qquad (n+1)/2 \leq i \leq n-1$

Said-Ball Curve: Recursive Algorithm 3

- If n is even,

$$v_{n/2} = v_{n/2}$$

$$v_i = (1-t)v_{n-i-1} + tv_{n-i}, \qquad i = \frac{n}{2}-1, \frac{n}{2}-2, \ldots, 1, 0$$

$$v_i = (1-t)v_i + tv_{i+1}, \qquad \text{otherwise.}$$

Then

- $\widehat{v}_i = v_i$ for $0 \leq i \leq n/2 - 1$
- $\widehat{v}_{i+1} = v_{i+1}$ for $n/2 \leq i \leq n - 1$

Said-Ball Curve: Recursive Algorithm 4

Then replace $\{v_i\}_0^{n-1}$ with $\{\widehat{v}_i\}_0^{n-1}$

Repeat the process until $n < 3$.

The value of $\mathbf{C}(t)$ corresponding to the value t is obtained as

$$\begin{aligned} a &= (1-t)\widehat{v}_0 + t\widehat{v}_1 \\ b &= (1-t)\widehat{v}_1 + t\widehat{v}_2 \\ \mathbf{C}(t) &= (1-t)a + tb \end{aligned}$$

Degree Elevation

Given the control points v_i for $i = 0, 1, \ldots, 2m+1$, the generalized Said-Ball curves of degree $(2m+1)$ can be expressed in terms of the generalized Said-Ball basis of degree $(2m+3)$ as follows:

$$\mathbf{C}(t) = \sum_{i=0}^{2m+3} v_i^1 S_i^{2m+3}$$

$$\begin{aligned}
v_0^{(1)} &= v_0 \\
v_{2m+3}^{(1)} &= v_{2m+1} \\
v_i^{(1)} &= \frac{i}{m+i+1} v_{i-1}^{(1)} + \frac{m+i}{m+i+1} v_i \\
v_{2m+3-i}^{(1)} &= \frac{i}{m+i+1} v_{2m+1-i}^{(1)} + \frac{m+i}{m+i+1} v_{2m+1-i}
\end{aligned}$$

$$v_{i+1}^{(1)} = v_{i+2}^{(1)} = \frac{1}{2}\left(v_m^{(1)} + v_{m+3}^{(1)}\right)$$

Wang-Ball Curve

Another generalization of Ball-curve is **Wang-Ball curve**. Given $n+1$ points $p_0, \ldots, p_n$, a Wang-Ball curve of degree n (order $n+1$) defined by these points, can be expressed as

$$\mathbf{W}(t) = \sum_{i=0}^{n} p_i A_i^n(t)$$

where

$$A_i^n(t) = \begin{cases} (2t)^i (1-t)^{i+2}, & 0 \le i \le \lfloor n/2 \rfloor - 1 \\ (2t)^{\lfloor n/2 \rfloor}(1-t)^{\lceil n/2 \rceil}, & i = \lfloor n/2 \rfloor \\ (2(1-t))^{\lfloor n/2 \rfloor} t^{\lceil n/2 \rceil}, & i = \lceil n/2 \rceil \\ A_{n-i}^n(1-t), & \lceil n/2 \rceil + 1 \le i \le n \end{cases}$$

Properties of Wang-Ball Curve

- Nonnegative basis functions
- Partition of Unity
- Convex Hull property: The Wang-Ball curve lies inside the convex hull formed by its control points.
- Two endpoints interpolation

Wang Algorithm

To compute the point on $\mathbf{W}(t)$ using the following steps:

It n is odd:

$$\widehat{p}_i = \begin{cases} p_i, & 0 \le i \le (n-3)/2 \\ (1-t)p_{(n-1)/2} + tp_{(n+1)/2}, & i = (n-1)/2 \\ p_{i+1}, & (n+1)/2 \le i \le n-1 \end{cases}$$

If n is even:

$$\widehat{p}_i = \begin{cases} p_i, & 0 \le i \le n/2-2 \\ (1-t)p_{n/2-1} + tp_{n/2}, & i = n/2-1 \\ (1-t)p_{n/2-1} + tp_{n/2+1}, & i = n/2 \\ p_{i+1}, & n/2+1 \le i \le n-1 \end{cases}$$

Wang Algorithm

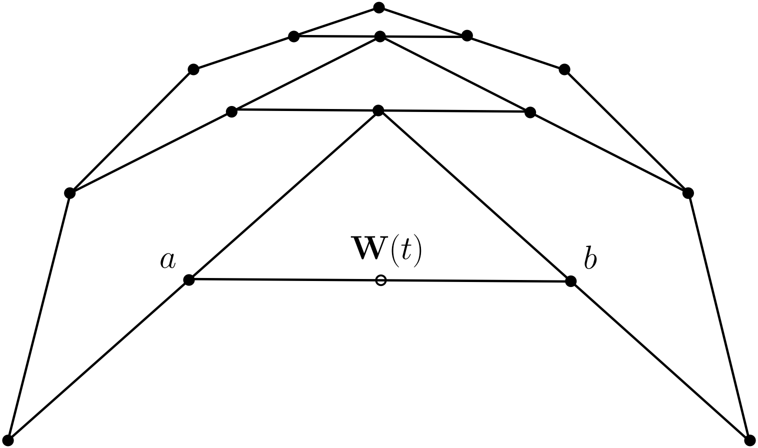
Then replace $\{p\}_0^{n-1}$ with $\{\widehat{p}\}_0^{n-1}$

Repeat the step until $n < 3$

The value of $\mathbf{W}(t)$ corresponding to the value of t is obtained by

$$\begin{aligned} a &= (1-t)\widehat{p}_0 + t\widehat{p}_1 \\ b &= (1-t)\widehat{p}_1 + t\widehat{p}_2 \\ \mathbf{W}(t) &= (1-t)a + tb \end{aligned}$$

Wang Algorithm: Graphical illustration



Wang Algorithm: Complexity

- For n is odd, this algorithm requires $(3n-1)/2$ additions and $3n-1$ multiplications
- For n is even, this algorithm requires $3n/2$ additions and $3n$ multiplications
- Computational complexity is $O(n)$

Wrapping up

So far we have seen many types of curves

1. Hermite curve
2. Bezier curve
3. Ball curve
   - Said-Ball curve
   - Wang-Ball curve

Next time we will compare and contrast them.