Concatenation of Matrix Transformations (Composition)

Concatenation of two 3D Translations

1) $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_{x_1} \\ 0 & 1 & 0 & t_{y_1} \\ 0 & 0 & 1 & t_{z_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_1} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2) $$\begin{bmatrix} x'' \\ y'' \\ z'' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_{x_2} \\ 0 & 1 & 0 & t_{y_2} \\ 0 & 0 & 1 & t_{z_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_2} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Concatenation of two 3D Translations (cont'd)

- Substituting 1 into 2:

$$\begin{bmatrix} x'' \\ y'' \\ z'' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_{x_2} \\ 0 & 1 & 0 & t_{y_2} \\ 0 & 0 & 1 & t_{z_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_{x_1} \\ 0 & 1 & 0 & t_{y_1} \\ 0 & 0 & 1 & t_{z_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_1} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x'' \\ y'' \\ z'' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_{x_2}+t_{x_1} \\ 0 & 1 & 0 & t_{y_2}+t_{y_1} \\ 0 & 0 & 1 & t_{z_2}+t_{z_1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- The result is a translation where the parameters are:

$$t_x = t_{x_1} + t_{x_2}$$
$$t_y = t_{y_1} + t_{y_2}$$
$$t_z = t_{z_1} + t_{z_2}$$

Scaling

$$S_1 = \begin{bmatrix} s_{x_1} & 0 & 0 & 0 \\ 0 & s_{y_1} & 0 & 0 \\ 0 & 0 & s_{z_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} s_{x_2} & 0 & 0 & 0 \\ 0 & s_{y_2} & 0 & 0 \\ 0 & 0 & s_{z_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_1 S_2 = \begin{bmatrix} s_{x_1}s_{x_2} & 0 & 0 & 0 \\ 0 & s_{y_1}s_{y_2} & 0 & 0 \\ 0 & 0 & s_{z_1}s_{z_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The result is a scaling where the scaling factors are :

$$s_x = s_{x_1} s_{x_1}$$
$$s_y = s_{y_1} s_{y_2}$$
$$s_z = s_{z_1} s_{z_2}$$

Rotation (2D)

1) $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2) $$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Rotation (2D) (cont'd)

- Substituting 1 into 2 :

$$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

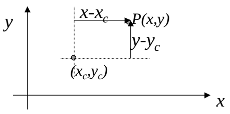
$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & 0 \\ \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The result is rotation with $\theta = \theta_1 + \theta_2$ ⇨

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation (2D) about a point

- Consider the rotation about (x_c, y_c)



- This is equivalent to:
  - 1) Translate coordinate system to new system with origin at (x_c, y_c) ⇨ $t_x = -x_c$, $t_y = -y_c$
  - 2) Rotate
  - 3) Translate Back , i.e.: ⇨ $t_x = x_c$, $t_y = y_c$

Rotation (2D) about a point (cont'd)

1) $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2) $$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

3) $$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}$$

Rotation (2D) about a point (cont'd)

- Substituting (1) into (2) into (3):

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \left(\underbrace{\begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}} \left(\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \left(\underbrace{\begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) \right) \right)$$

Rotation (2D) about a point (cont'd)

- Now, matrix multiplication is *associative* (but not commutative - *eg*: rotation followed by translation is not equivalent to vice versa, reversing order of rotations changes the result)

Rotation (2D) about a point (cont'd)

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & -x_c\cos\theta + y_c\sin\theta + x_c \\ \sin\theta & \cos\theta & -x_c\sin\theta - y_c\cos\theta + y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation (2D) about a point (concl'd)

$$= \begin{bmatrix} x\cos\theta - y\sin\theta - x_c\cos\theta + y_c\sin\theta + x_c \\ x\sin\theta + y\cos\theta - x_c\sin\theta - y_c\cos\theta + y_c \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (x - x_c)\cos\theta - (y - y_c)\sin\theta + x_c \\ (x - x_c)\sin\theta + (y - y_c)\cos\theta + y_c \\ 1 \end{bmatrix}$$

Thus:

$$x''' = (x - x_c)\cos\theta - (y - y_c)\sin\theta + x_c$$
$$y''' = (x - x_c)\sin\theta + (y - y_c)\cos\theta + y_c$$

Scaling about arbitrary point

- Similarly, can scale around an arbitrary point (eg: center of an object) (x_c,y_c)
- Translate coordinate system to new system with origin at (x_c,y_c)
  - i.e.: $t_x=-x_c \quad t_y=-y_c$
- Scale
- Translate back
  - i.e.: $t_x=x_c \quad t_y = y_c$

Scaling about arbitrary point (cont'd)

$$\begin{bmatrix}1 & 0 & x_c\\0 & 1 & y_c\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}s_x & 0 & 0\\0 & s_y & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -x_c\\0 & 1 & -y_c\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & x_c\\0 & 1 & y_c\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}s_x & 0 & -x_c s_x\\0 & s_y & -y_c s_y\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}s_x & 0 & x_c - x_c s_x\\0 & s_y & y_c - y_c s_y\\0 & 0 & 1\end{bmatrix}$$

Composition

- Thus, concatenation of matrix transformations enabled us to solve a more general problem in terms of simple matrix transformations
- But there is another important observation to be made. Equations represent the desired transformations in single *compound* transformation matrices

Composition (cont'd)

- Any number of simple transformations can be concatenated into a single compound transformation matrix
- Q: Why is this important ?
- A: Because it essentially means that complicated transformations require virtually the same amount of computation as simple ones

Composition (concl'd)

- Instead of performing each transformation on all points, concatenate the matrices, and apply the compound transformation matrix to all points
- This also saves storage, since it is not necessary to keep all the simple transformation matrices. All that is needed is the current compound transformation matrix

Inverse Transformation

- Transformation which cancels the effect of the original
- Inverse transformation described by inverse matrix
- From geometrical intuition, it is easy to establish inverses

Scaling Inverse

$$S^{-1} = \begin{bmatrix} 1/s_x & 0 & 0 & 0 \\ 0 & 1/s_y & 0 & 0 \\ 0 & 0 & 1/s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation Inverse

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Inverse

- Inverse of a rotation matrix is simply the rotation matrix with the angle replaced by an angle of same magnitude but opposite sign:

$$R^{-1} = R(-\theta)$$

Orthogonal Matrix

- Two vectors are *orthogonal* if their inner product is zero
- Set of vectors is *orthogonal if all* vectors are pairwise orthogonal
- Square matrix is orthogonal if its column vectors are orthogonal and all column vectors are of unit length

Orthogonal Matrix (cont'd)

- Inverse of an orthogonal matrix is its transpose

$$R^{-1} = R^t$$

- This provides a quick method for obtaining the inverse of R, since rotation matrices are orthogonal matrices

Matrix for Multiple Rotation About Different Axes

$$R_{yx}(\phi,\theta) = R_y(\phi)R_x(\theta) = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & \cos\theta\sin\phi & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ -\sin\phi & \cos\phi\sin\theta & \cos\phi\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(rotate about x then y)

$$R_{xy}(\theta,\phi) = R_x(\theta)R_y(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta & -\cos\phi\sin\theta & 0 \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(rotate about y then x)

Matrix for Multiple Rotation About Different Axes (cont'd)

- Observe that the resulting matrices are different
- This is due to the fact that rotation is not commutative. Thus, there is no single matrix which embodies all the rotation matrices

Matrix for Multiple Rotation About Different Axes (concl'd)

- It is interesting to observe that one matrix can be obtained from the other by changing the signs of the angles and taking the transpose. To see why:

$$\begin{aligned} R_{yx} &= R_y(\phi)R_x(\theta) \\ &= (R^{-1}{}_x(\theta)R^{-1}{}_y(\phi))^{-1} \\ &= (R_x(-\theta)R_y(-\phi))^{-1} \\ &= (R_x(-\theta)R_y(-\phi))^t \end{aligned}$$

Rotation (3D)

- The last step requires that we show that

$$(R_x(-\theta)R_y(-\phi))(R_x(-\theta)R_y(-\phi))^t = I$$

- form

$$R_x(-\theta)R_y(-\phi)R^t{}_y(-\phi)R^t{}_x(-\theta) = I$$

$$R_x(-\theta)\underbrace{R_y(-\phi)R^t{}_y(-\phi)}_{I}R^t{}_x(-\theta) = I$$

$$\underbrace{R_x(-\theta)R^t{}_x(-\theta)}_{I} = I$$